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Positive solutions of the Dirichlet problem for the prescribed mean curvature equation

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ABSTRACT

We discuss existence and multiplicity of positive solutions of the prescribed mean curvature problem

$$-\operatorname{div}(\nabla u / \sqrt{1 + |\nabla u|^2}) = \lambda f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

in a general bounded domain $\Omega \subset \mathbb{R}^N$, depending on the behavior at zero or at infinity of $f(x, s)$, or of its potential $F(x, s) = \int_0^s f(x, t) dt$. Our main effort here is to describe, in a way as exhaustive as possible, all configurations of the limits of $F(x, s)/s^2$ at zero and of $F(x, s)/s$ at infinity, which yield the existence of one, two, three or infinitely many positive solutions. Either strong, or weak, or bounded variation solutions are considered. Our approach is variational and combines critical point theory, the lower and upper solutions method and elliptic regularization.

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1. Introduction

Let us consider the Dirichlet problem for the prescribed mean curvature equation

$$\begin{cases} -\operatorname{div}(\nabla u / \sqrt{1 + |\nabla u|^2}) = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

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where Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{0,1}$ boundary $\partial\Omega$, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function and $\lambda > 0$ is a real parameter. The potential F of f is defined by

$$F(x, s) = \int_0^s f(x, t) dt.$$

Existence, non-existence and multiplicity of positive solutions of problem (1) have been discussed by several authors in the last decades. The case where Ω is a ball and u is a classical radially symmetric solution has been studied, among others, by Ni and Serrin [31–33], Serrin [42], Peletier and Serrin [40], Atkinson, Peletier and Serrin [2], Ishimura [23], Kusano and Swanson [24], Clément, Manásevich and Mitidieri [10], Franchi, Lanconelli and Serrin [16], Bidaut-Veron [3], Conti and Gazzola [12], Chang and Zhang [8], del Pino and Guerra [15], also in relation with the existence of ground states. The one-dimensional problem has been rather thoroughly discussed in a series of recent papers by Bonheure, Habets, Obersnel and Omari [22,5,35,6], Bereanu and Mawhin [4] and Pan [39]. The case where Ω displays no special symmetry, $f(x, s)$ behaves like a power s^p ($p > 0$) and u is either a classical, or a weak, or a bounded variation solution, has been considered by Nakao [30], Coffman and Ziemer [11], Noussair, Swanson and Yang [34], Habets and Omari [21], Le [25,27]. It is worthwhile to mention that the case where the domain Ω has no special symmetry and the searched solution u (classical or bounded variation) has not a prescribed sign has been the subject of deep studies in the classical works of De Giorgi, Serrin, Federer, Finn, Miranda, Giusti, Giaquinta, Trudinger, Ladyzenskaia, Ural'tseva, Temam, Gerhard, Simon et al.

In this paper we deal with positive bounded variation solutions of (1) in a genuine partial differential equation setting, i.e. in space dimension $N \geq 2$, and in fact one of our aims here is to extend to higher dimensions the results obtained in [5]. However our conclusions are still valid, and sometimes even new, in the one-dimensional case.

A *bounded variation solution* of (1) is a function $u \in BV(\Omega)$ such that $f(\cdot, u) \in L^p(\Omega)$, for some $p > N$, and

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \lambda \int_{\Omega} f(x, u)(v - u) dx \quad (2)$$

for every $v \in BV(\Omega)$. The functional $\mathcal{J} : BV(\Omega) \rightarrow \mathbb{R}$ is defined by

$$\mathcal{J}(w) = \int_{\Omega} \sqrt{1 + |Dw|^2} + \int_{\partial\Omega} |w|_{\partial\Omega} d\mathcal{H}_{N-1},$$

where, for any $w \in BV(\Omega)$,

$$\int_{\Omega} \sqrt{1 + |Dw|^2} = \sup \left\{ \int_{\Omega} \left(w \sum_{i=1}^N \frac{\partial z_i}{\partial x_i} + z_{N+1} \right) dx \mid z_i \in C_0^1(\Omega) \right. \\ \left. \text{for } i = 1, 2, \dots, N+1 \text{ and } \left\| \sum_{i=1}^{N+1} z_i^2 \right\|_{L^\infty(\Omega)} \leq 1 \right\},$$

$w|_{\partial\Omega} \in L^1(\partial\Omega, \mathcal{H}_{N-1})$ is the trace of w on $\partial\Omega$ and \mathcal{H}_{N-1} denotes the $(N-1)$ -dimensional Hausdorff measure.

This notion of bounded variation solution is equivalent (see Remark 2.1) to requiring that $u \in BV(\Omega)$ satisfies the Euler equation

$$\int_{\Omega} \frac{(Du)^a (Dv)^a}{\sqrt{1 + |(Du)^a|^2}} dx + \int_{\Omega} \frac{Du}{|Du|} \frac{Dv}{|Dv|} d|Dv|^s + \int_{\partial\Omega} \operatorname{sgn}(u)|_{\partial\Omega} v|_{\partial\Omega} d\mathcal{H}_{N-1} = \lambda \int_{\Omega} f(x, u) v dx$$

for every $v \in BV(\Omega)$ such that $|Dv|^s$ is absolutely continuous with respect to $|Du|^s$ and $v|_{\partial\Omega}(x) = 0$ \mathcal{H}_{N-1} -a.e. on the set $\{x \in \partial\Omega \mid u|_{\partial\Omega}(x) = 0\}$. Here, for $w \in BV(\Omega)$, $Dw = (Dw)^a + (Dw)^s$ is the Lebesgue decomposition of the measure Dw in its absolutely continuous part and its singular part with respect to the N -dimensional Lebesgue measure in \mathbb{R}^N , $|Dw|$ denotes the total variation of the measure Dw , $|Dw| = |Dw|^a + |Dw|^s$ is the Lebesgue decomposition of $|Dw|$ and $\frac{Dw}{|Dw|}$ is the density of Dw with respect to its total variation $|Dw|$.

Of course, if a bounded variation solution u of (1) is more regular, then it is a solution in some stronger sense. For instance, if $u \in W_0^{1,1}(\Omega)$, then (see Remark 2.3) it is a *weak solution* of (1), in the sense that

$$\int_{\Omega} \frac{\nabla u \nabla v}{\sqrt{1 + |\nabla u|^2}} dx = \lambda \int_{\Omega} f(x, u) v dx$$

for every $v \in W_0^{1,1}(\Omega)$. If in addition $u \in W^{2,p}(\Omega)$ for some $p > N$, then (see Remark 2.4) u is a *strong solution* of (1), in the sense that

$$-\operatorname{div}(\nabla u / \sqrt{1 + |\nabla u|^2})(x) = \lambda f(x, u(x)) \quad \text{a.e. in } \Omega, \quad u(x) = 0 \quad \text{on } \partial\Omega.$$

Throughout this paper by a *solution* of (1), without any further specification, we will always mean a bounded variation solution.

We also say that a solution u of (1) is *positive* if $u(x) \geq 0$ for a.e. $x \in \Omega$ and $u(x) > 0$ on a set of positive measure.

Our study of problem (1) will essentially rely on variational methods. This is a quite natural approach because u is a bounded variation solution of (1) if and only if 0 is a subgradient at u of the action functional $\mathcal{I}_{\lambda} : BV(\Omega) \rightarrow \mathbb{R}$ defined by

$$\mathcal{I}_{\lambda}(v) = \mathcal{J}(v) - \lambda \int_{\Omega} F(x, v) dx.$$

We will infer the existence of positive solutions of (1) by comparing $F(x, s)$ with s^2 near zero and with s at infinity. This is suggested by the fact that the curvature operator $\operatorname{div}(\nabla u / \sqrt{1 + |\nabla u|^2})$ behaves like the Laplace operator $\Delta u = \operatorname{div}(\nabla u)$ near zero and like the 1-Laplace operator $\Delta_1 u = \operatorname{div}(\nabla u / |\nabla u|)$ at infinity. Some specific configurations of the limits of $F(x, s)/s^2$ at 0 and of $F(x, s)/s$ at $+\infty$ will then yield the existence of one, two, three, or infinitely many positive solutions of (1), thus reproducing the multiplicity phenomena already pointed out in [5] in the one-dimensional case. The study of problems where the differential operators exhibit different degrees of homogeneity at zero and at infinity seems to have been little studied in the literature: we refer to [9,17,35] for some recent contributions in this direction.

In order to minimize technicalities and to describe, in the course of this introduction, our main results in a neat and simple way, we will discuss in the sequel some model situations where $f(x, s)$ is independent of x , i.e. $f(x, s) = f(s)$ in $\Omega \times \mathbb{R}$, and its potential $F(s) = \int_0^s f(t) dt$ behaves like a power s^{p_0} in a neighborhood of 0, or a power $s^{p_{\infty}}$ in a neighborhood of $+\infty$. Additional smoothness of $\partial\Omega$ will be sometimes assumed. The statements produced below, which are at least partially known, will be generalized in manifold directions in Section 3. We remark that the hypotheses here considered are put on the potential F just to facilitate a comparison with the results given in Section 3, even though all assumptions could have been expressed in an equivalent way directly on f .

We start considering conditions that yield the existence of at least one positive solution. First we discuss the case where the potential is subquadratic at 0 or sublinear at $+\infty$.

Potential subquadratic at 0. Suppose that Ω has a $C^{1,1}$ boundary $\partial\Omega$. Assume that there exist $p_0 \in [1, 2[$ and $s_0 > 0$ such that $F(s) = s^{p_0}$ for every $s \in [0, s_0]$. Then there exists $\lambda^* \in]0, +\infty[$ such that, for every $\lambda \in]0, \lambda^*[$, problem (1) has at least one positive weak solution.

Potential sublinear at $+\infty$. Suppose that $f(0) \geq 0$. Assume that there exist $p_\infty \in]-\infty, 1[$ and $s_\infty > 0$ such that, up to an additive constant, $F(s) = s^{p_\infty}$ for every $s \in [s_\infty, +\infty[$. Then there exists $\lambda_* \in [0, +\infty[$ such that, for every $\lambda \in]\lambda_*, +\infty[$, problem (1) has at least one positive solution.

Potential subquadratic at 0 and sublinear at $+\infty$. Assume that there exist $p_0 \in [1, 2[$ and $p_\infty \in]-\infty, 1[$ and $s_0, s_\infty > 0$ such that $F(s) = s^{p_0}$ for every $s \in [0, s_0]$ and, up to an additive constant, $F(s) = s^{p_\infty}$ for every $s \in [s_\infty, +\infty[$. Then, for every $\lambda > 0$, problem (1) has at least one positive solution.

The proof of the first result is worked out by minimizing the action functional associated with a suitably modified problem, which is uniformly elliptic, and on gradient estimates for the corresponding solutions. Whereas, the last two statements are obtained by a direct minimization in $BV(\Omega)$ of the functional \mathcal{I}_λ associated with (1). Related results can be found in [21,25,27,36]. A class of model functions to which these three statements apply are

$$f(s) = \min\{(s^+)^{p_0-1}, (s^+)^{p_\infty-1}\},$$

with $p_0 \in [1, 2[$ and $p_\infty \in]-\infty, 1[$.

In the limiting cases where $p_0 = 2$ and $p_\infty = 1$, we can prove some sharper results involving the principal eigenvalues λ_1 and μ_1 of $-\Delta$ and $-\Delta_1$ with Dirichlet boundary conditions; we refer to Section 2 for the definition of these and other related spectral-type constants. In particular the following result holds.

Potential quadratic at 0 and linear at $+\infty$. Assume that there exist constants $\kappa_0 \in [\lambda_1, +\infty[$, $\kappa_\infty \in]0, \mu_1[$ and $s_0, s_\infty > 0$ such that $F(s) = \frac{1}{2}\kappa_0 s^2$ for every $s \in [0, s_0]$ and, up to an additive constant, $F(s) = \kappa_\infty s$ for every $s \in [s_\infty, +\infty[$. Then, for $\lambda = 1$, problem (1) has at least one positive solution.

Next we consider the case where the potential is superquadratic at 0 or superlinear at $+\infty$.

Potential superquadratic at 0. Suppose that Ω has a $C^{1,1}$ boundary $\partial\Omega$. Assume that there exist $p_0 \in]2, 2^*[$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ if $N = 2$, and $s_0 > 0$ such that $F(s) = s^{p_0}$ for every $s \in [0, s_0]$. Then there exists $\lambda_* \in [0, +\infty[$ such that, for every $\lambda \in]\lambda_*, +\infty[$, problem (1) has at least one positive strong solution.

Potential superlinear at $+\infty$. Suppose that $f(0) \geq 0$. Assume that there exist $p_\infty \in]1, 1^*[$, where $1^* = \frac{N}{N-1}$, and $s_\infty > 0$ such that, up to an additive constant, $F(s) = s^{p_\infty}$ for every $s \in [s_\infty, +\infty[$. Then there exists $\lambda^* \in]0, +\infty[$ such that, for every $\lambda \in]0, \lambda^*[$, problem (1) has at least one positive solution.

The former result was obtained in [11] by a clever adaptation to problem (1) of the Nehari method; a hopefully more transparent proof, based on the same approach, is given in [37]. The latter result is obtained by an elliptic regularization procedure similar to that performed in [5] for the one-dimensional case. We remark that approximating the non-uniformly elliptic problem (1), by adding the term $-\varepsilon \Delta u$, has been repeatedly used in the literature, starting with [43], when $f(x, s) = f(x)$ does not depend on s . What is new here is to adopt this technique to deal with functions $f(s)$, or more generally $f(x, s)$, which behave as s^{p-1} for some $p > 1$ at $+\infty$, and to replace the perturbation $-\varepsilon \Delta u$ with $-\varepsilon \Delta_r = -\varepsilon \operatorname{div}(|\nabla u|^{r-2} \nabla u)$, where r is chosen such that $1 < r < p$ in order to preserve the mountain pass geometry of the original functional \mathcal{I}_λ for small values of $\lambda > 0$. Each approximating problem is then solved in $W_0^{1,r}(\Omega)$ and the obtained solutions are controlled by suitable

$W^{1,1}$ -estimates, which allow to get a bounded variation solution of (1) by passing to the limit. Related results can be found in [25,28], where non-smooth critical point theory or finite-dimensional approximation were respectively used.

Just combining the two preceding results yields the following statement, which however has an important drawback: unlike the corresponding one-dimensional case (cf. [5, Theorem 3.5]) we are unable here to prove the existence of a solution for any given $\lambda > 0$; it remains an open problem to fill this gap.

Potential superquadratic at 0 and superlinear at $+\infty$. Suppose that Ω has a $C^{1,1}$ boundary $\partial\Omega$. Assume that there exist $p_0 \in]2, 2^*[$, $p_\infty \in]1, 1^*[$ and $s_0, s_\infty > 0$ such that $F(s) = s^{p_0}$ for every $s \in [0, s_0]$ and, up to an additive constant, $F(s) = s^{p_\infty}$ for every $s \in [s_\infty, +\infty[$. Then there exist $\lambda_* \in [0, +\infty[$ and $\lambda^* \in]0, +\infty[$ such that, for every $\lambda \in]0, \lambda^*[\cup]\lambda_*, +\infty[$, problem (1) has at least one positive solution.

These last three statements apply to functions like

$$f(s) = \min\{(s^+)^{p_0-1}, (s^+)^{p_\infty-1}\},$$

with $p_0 \in]2, 2^*[$ and $p_\infty \in]1, 1^*[$.

Subquadraticity, or respectively superquadraticity, at zero can be combined with superlinearity, or respectively sublinearity, at infinity to produce multiplicity of solutions. A result substantially similar to our first statement below was previously obtained in [25] by a different approach based on non-smooth critical point theory. On the contrary our second result below is new in dimensions larger than one.

Potential subquadratic at 0 and superlinear at $+\infty$. Suppose that Ω has a $C^{1,1}$ boundary $\partial\Omega$. Assume that there exist $p_0 \in [1, 2[$, $p_\infty \in]1, 1^*[$ and $s_0, s_\infty > 0$ such that $F(s) = s^{p_0}$ for every $s \in [0, s_0]$ and, up to an additive constant, $F(s) = s^{p_\infty}$ for every $s \in [s_\infty, +\infty[$. Then there exists $\lambda^* \in]0, +\infty[$ such that, for every $\lambda \in]0, \lambda^*[$, problem (1) has at least two positive solutions, one of which is weak.

This result applies for instance to

$$f(s) = (s^+)^{p_0-1},$$

with $p_0 \in]1, 1^*[$.

Potential superquadratic at 0 and sublinear at $+\infty$. Suppose that Ω has a $C^{1,1}$ boundary $\partial\Omega$. Assume that there exist $p_0 \in]2, 2^*[$, $p_\infty \in]-\infty, 1[$ and $s_0, s_\infty > 0$ such that $F(s) = s^{p_0}$ for every $s \in [0, s_0]$ and, up to an additive constant, $F(s) = s^{p_\infty}$ for every $s \in [s_\infty, +\infty[$. Then there exists $\lambda_* \in [0, +\infty[$ such that, for every $\lambda \in]\lambda_*, +\infty[$, problem (1) has at least two positive solutions, one of which is strong.

Functions that can be considered here are

$$f(s) = \min\{(s^+)^{p_0-1}, (s^+)^{p_\infty-1}\},$$

with $p_0 \in]2, 2^*[$ and $p_\infty \in]-\infty, 1[$.

In the case where the potential is superquadratic at 0 and superlinear at $+\infty$, the introduction of a second parameter allows to get the existence of three positive solutions. Namely, let us consider the model two-parameter problem

$$\begin{cases} -\operatorname{div}(\nabla u / \sqrt{1 + |\nabla u|^2}) = \min\{\lambda(u^+)^{p_\infty-1}, \mu(u^+)^{p_0-1}\} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3)$$

with $p_\infty > 1$ and $p_0 > 2$. A careful analysis of the geometric features of the associated action functional leads to the following result.

Potential superquadratic at 0 and superlinear at $+\infty$ depending on two parameters. Suppose that Ω has a $C^{1,1}$ boundary $\partial\Omega$. Assume that $p_0 \in]2, 2^*[$ and $p_\infty \in]1, 1^*[$. Then there exist $\lambda^* \in]0, +\infty]$ and a function $\mu_* :]0, \lambda^*[\rightarrow [0, +\infty[$ such that, for every $\lambda \in]0, \lambda^*[$ and $\mu \in]\mu_*(\lambda), +\infty[$, problem (3) has at least three positive solutions, two of which are weak.

We conclude this overview by observing that we can also deal with cases where the potential is neither subquadratic nor superquadratic at zero and it is neither sublinear nor superlinear at infinity, but it oscillates in between. In this frame we can establish the existence of infinitely many positive solutions. The proof combines the lower and upper solutions method, local minimization and critical values estimates; some ideas from [38,21,36] are exploited too.

Potential oscillatory at 0. Suppose that Ω has a $C^{1,1}$ boundary $\partial\Omega$. Assume that $\liminf_{s \rightarrow 0^+} F(s)/s^2 = 0$ and $\limsup_{s \rightarrow 0^+} F(s)/s^2 = +\infty$. Then, for every $\lambda > 0$, problem (1) has an infinite sequence of positive weak solutions.

Potential oscillatory at $+\infty$. Assume that $\liminf_{s \rightarrow +\infty} F(s)/s = 0$ and $\limsup_{s \rightarrow +\infty} F(s)/s = +\infty$. Then, for every $\lambda > 0$, problem (1) has an infinite sequence of positive solutions.

The rest of this paper is organized as follows: in Section 2 we collect some basic definitions and statements and in Section 3 we state and prove our existence and multiplicity results.

2. Preliminaries

In this section we list some notation, definitions and facts that will be used in the sequel.

Bounded variation function. Let Ω be an open set in \mathbb{R}^N ($N \geq 2$). For any $u \in L^1(\Omega)$ we put

$$\int_{\Omega} |Du| = \sup \left\{ \int_{\Omega} u \sum_{i=1}^N \frac{\partial v_i}{\partial x_i} dx \mid v_i \in C_0^1(\Omega) \text{ for } i = 1, 2, \dots, N \text{ and } \left\| \sum_{i=1}^N v_i^2 \right\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

A function $u \in L^1(\Omega)$ is said to have bounded variation in Ω if $\int_{\Omega} |Du| < +\infty$ (see [20, p. 3]). The linear space of all functions having bounded variation in Ω is denoted by $BV(\Omega)$.

Poincaré inequality (see [29, p. 228], [20, p. 24]). Assume

(h_1) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{0,1}$ boundary $\partial\Omega$.

For each $p \in [1, \frac{N}{N-1}]$ there exists a constant $\mu_p > 0$ such that

$$\mu_p \left(\int_{\Omega} |u|^p dx \right)^{\frac{1}{p}} \leq \left(\int_{\Omega} |Du| + \int_{\partial\Omega} |u|_{\partial\Omega} d\mathcal{H}_{N-1} \right) \quad (4)$$

for every $u \in BV(\Omega)$, where $u|_{\partial\Omega} \in L^1(\partial\Omega, \mathcal{H}_{N-1})$ is the trace of u on $\partial\Omega$ (see [20, p. 37]) and \mathcal{H}_{N-1} denotes the $(N-1)$ -dimensional Hausdorff measure.

The space $BV(\Omega)$. Assume (h_1). The space $BV(\Omega)$, equipped with the norm

$$\|u\|_{BV(\Omega)} = \int_{\Omega} |Du| + \int_{\partial\Omega} |u|_{\partial\Omega} d\mathcal{H}_{N-1},$$

is a Banach space continuously embedded into $L^{\frac{N}{N-1}}(\Omega)$ and compactly embedded into $L^q(\Omega)$ for any $q \in [1, \frac{N}{N-1}[$ (see [20, pp. 24, 17]).

Area functional. Assume (h_1) . The area functional $\mathcal{J} : BV(\Omega) \rightarrow \mathbb{R}$ is defined by

$$\mathcal{J}(u) = \int_{\Omega} \sqrt{1 + |Du|^2} + \int_{\partial\Omega} |u|_{\partial\Omega} d\mathcal{H}_{N-1},$$

where

$$\int_{\Omega} \sqrt{1 + |Du|^2} = \sup \left\{ \int_{\Omega} \left(u \sum_{i=1}^N \frac{\partial v_i}{\partial x_i} + v_{N+1} \right) dx \mid \right. \\ \left. v_i \in C_0^1(\Omega) \text{ for } i = 1, 2, \dots, N+1 \text{ and } \left\| \sum_{i=1}^{N+1} v_i^2 \right\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

The functional \mathcal{J} is convex and, also by the continuity of the trace map (see [20, Theorem 2.11]), is Lipschitz continuous in $BV(\Omega)$ and lower semicontinuous in $BV(\Omega)$ with respect to the L^1 -topology (see [20, p. 160]). Note that, if $u \in W_0^{1,1}(\Omega)$, then

$$\mathcal{J}(u) = \int_{\Omega} \sqrt{1 + |\nabla u|^2} dx$$

and the restriction of \mathcal{J} to $W_0^{1,1}(\Omega)$ is Gateaux differentiable at any point $u \in W_0^{1,1}(\Omega)$, with

$$\mathcal{J}'(u)(w) = \int_{\Omega} \frac{\nabla u \nabla w}{\sqrt{1 + |\nabla u|^2}} dx$$

for every $w \in W_0^{1,1}(\Omega)$.

Approximation property (see [1, pp. 491, 498], [13, Proposition 2], [26, Lemma 2.3]). Assume (h_1) . For any $u \in BV(\Omega)$ and $q \in [1, \frac{N}{N-1}[$, there exists a sequence $(u_n)_n \subset W_0^{1,1}(\Omega)$ such that

$$\lim_{n \rightarrow +\infty} u_n = u \quad \text{in } L^q(\Omega), \\ \lim_{n \rightarrow +\infty} \int_{\Omega} |\nabla u_n| dx = \int_{\Omega} |Du| + \int_{\partial\Omega} |u|_{\partial\Omega} d\mathcal{H}_{N-1}, \\ \lim_{n \rightarrow +\infty} \mathcal{J}(u_n) = \mathcal{J}(u).$$

Lattice property (see [36, Proposition 2.2]). Assume (h_1) . For any $u, v \in BV(\Omega)$, we have $\min\{u, v\}, \max\{u, v\} \in BV(\Omega)$ and

$$\mathcal{J}(\min\{u, v\}) + \mathcal{J}(\max\{u, v\}) \leq \mathcal{J}(u) + \mathcal{J}(v). \quad (5)$$

Bounded variation solution. Assume (h_1) . A function $u \in BV(\Omega)$ is said to be a bounded variation solution of (1), for a given $\lambda > 0$, if $f(\cdot, u) \in L^p(\Omega)$ for some $p > N$, and

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \lambda \int_{\Omega} f(x, u)(v - u) dx$$

for every $v \in BV(\Omega)$. As we already pointed out, by a solution of (1) in this work we always mean a bounded variation solution.

Remark 2.1. Assume (h_1) ,

(h_2) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions, i.e. for a.e. $x \in \Omega$, $f(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and, for every $s \in \mathbb{R}$, $f(\cdot, s) : \Omega \rightarrow \mathbb{R}$ is measurable

and

(h_3) there exist constants $q \in]1, \frac{N}{N-1}[$, $c_1 > 0$ and a function $c_2 \in L^{\frac{q}{q-1}}(\Omega)$ such that

$$|f(x, s)| \leq c_1 |s|^{q-1} + c_2(x)$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.

Let $\mathcal{F} : L^q(\Omega) \rightarrow \mathbb{R}$ be the potential functional defined by

$$\mathcal{F}(v) = \int_{\Omega} F(x, v) dx.$$

It follows from [14, Theorem 2.8] that \mathcal{F} is of class C^1 . For each $\lambda > 0$ we define the action functional $\mathcal{I}_{\lambda} : BV(\Omega) \rightarrow \mathbb{R}$ by

$$\mathcal{I}_{\lambda}(v) = \mathcal{J}(v) - \lambda \mathcal{F}(v).$$

Note that \mathcal{I}_{λ} is lower semicontinuous in $BV(\Omega)$ with respect to the L^q -topology. According to the convexity and the continuity of \mathcal{J} and the differentiability of \mathcal{F} in $BV(\Omega)$, it is natural to say that a function $u \in BV(\Omega)$ is a solution of (1) if $0 \in \partial \mathcal{J}(u) - \lambda \mathcal{F}'(u)$, i.e. $\lambda \mathcal{F}'(u) \in \partial \mathcal{J}(u)$, where $\partial \mathcal{J}(u)$ denotes the subdifferential of \mathcal{J} at u . This means that

$$\mathcal{J}(v) \geq \mathcal{J}(u) + \lambda \mathcal{F}'(u)(v - u),$$

that is (2) holds for every $v \in BV(\Omega)$. Note also that u is a bounded variation solution of (1) if and only if u minimizes in $BV(\Omega)$ the functional $\mathcal{H}_{\lambda, u} : BV(\Omega) \rightarrow \mathbb{R}$ defined by

$$\mathcal{H}_{\lambda, u}(v) = \mathcal{J}(v) - \lambda \mathcal{F}'(u)(v). \quad (6)$$

According to [1, Section 3] this is equivalent to saying that $u \in BV(\Omega)$ satisfies the Euler equation

$$\int_{\Omega} \frac{(Du)^a (Dv)^a}{\sqrt{1 + |(Du)^a|^2}} dx + \int_{\Omega} \frac{Du}{|Du|} \frac{Dv}{|Dv|} d|Dv|^s + \int_{\partial\Omega} \text{sgn}(u) v_{|\partial\Omega} d\mathcal{H}_{N-1} = \lambda \int_{\Omega} f(x, u) v dx \quad (7)$$

for every $v \in BV(\Omega)$ such that $|Dv|^s$ is absolutely continuous with respect to $|Du|^s$ and $v_{|\partial\Omega}(x) = 0$ \mathcal{H}_{N-1} -a.e. on the set $\{x \in \partial\Omega \mid u_{|\partial\Omega}(x) = 0\}$. Here, for $w \in BV(\Omega)$, $Dw = (Dw)^a dx + (Dw)^s$ is the Lebesgue decomposition of the measure Dw in its absolutely continuous part and its singular part with respect to the N -dimensional Lebesgue measure in \mathbb{R}^N , $|Dw|$ denotes the total variation of the measure Dw , $|Dw| = |Dw|^a dx + |Dw|^s$ is the Lebesgue decomposition of $|Dw|$ and $\frac{Dw}{|Dw|}$ is the density of Dw with respect to its total variation $|Dw|$. Eq. (7) yields an alternative formulation of the notion of bounded variation solution of (1) we have previously introduced.

Remark 2.2. Assume (h_1) , (h_2) and (h_3) . If $u \in BV(\Omega)$ is a local minimizer of the functional \mathcal{I}_λ , then u satisfies (2) for every $v \in BV(\Omega)$. Indeed, as \mathcal{J} is convex and \mathcal{F} is of class C^1 , we have, for each $v \in BV(\Omega)$ sufficiently close to u and every $t \in]0, 1[$,

$$J(u) - \lambda \mathcal{F}(u) \leq (1-t)J(u) + tJ(v) - \lambda \mathcal{F}(u) - \lambda \left(\int_0^1 \mathcal{F}'(u + st(v-u)) ds \right) t(v-u).$$

Hence, rearranging and dividing by $t > 0$, we get

$$J(u) - J(v) \leq \lambda \left(\int_0^1 \mathcal{F}'(u + st(v-u)) ds \right) (u-v)$$

and, letting $t \rightarrow 0^+$,

$$J(u) - \lambda \mathcal{F}'(u)(u) \leq J(v) - \lambda \mathcal{F}'(u)(v).$$

This means that the functional $\mathcal{H}_{\lambda,u}$ defined by (6) has a local minimum at u . As $\mathcal{H}_{\lambda,u}$ is convex, u is a global minimizer of $\mathcal{H}_{\lambda,u}$ and then (2) holds for every $v \in BV(\Omega)$.

Weak solution. Assume (h_1) . A function $u \in W_0^{1,1}(\Omega)$ is said to be a weak solution of (1), for a given $\lambda > 0$, if $f(\cdot, u) \in L^p(\Omega)$, for some $p > N$, and

$$\int_{\Omega} \frac{\nabla u \nabla v}{\sqrt{1 + |\nabla u|^2}} dx = \lambda \int_{\Omega} f(x, u) v dx \quad (8)$$

for every $v \in W_0^{1,1}(\Omega)$.

Strong solution. Assume (h_1) . A function $u \in W^{2,p}(\Omega)$, for some $p > N$, is said to be a strong solution of (1), for a given $\lambda > 0$, if

$$-\operatorname{div}(\nabla u / \sqrt{1 + |\nabla u|^2})(x) = \lambda f(x, u(x)) \quad \text{a.e. in } \Omega, \quad u(x) = 0 \quad \text{on } \partial\Omega. \quad (9)$$

Positive solution. A solution u of (1) is said to be *positive* if $u(x) \geq 0$ a.e. in Ω and $u(x) > 0$ in a set of positive measure.

Remark 2.3. A weak solution u of (1) is a bounded variation solution of (1). Indeed, as the restriction to $W_0^{1,1}(\Omega)$ of the functional \mathcal{J} is convex and Gateaux differentiable, we have

$$\lambda \int_{\Omega} f(x, u)(v-u) dx = \int_{\Omega} \frac{\nabla u \nabla (v-u)}{\sqrt{1 + |\nabla u|^2}} dx = \mathcal{J}'(u)(v-u) \leq \mathcal{J}(v) - \mathcal{J}(u)$$

for every $v \in W_0^{1,1}(\Omega)$. The above stated approximation property in $BV(\Omega)$ implies that (2) holds for every $v \in BV(\Omega)$.

Conversely, a bounded variation solution u of (1), with $u \in W_0^{1,1}(\Omega)$, is a weak solution of (1). Indeed, fix $w \in W_0^{1,1}(\Omega)$. From (2) we have, for every $t \neq 0$,

$$\operatorname{sgn}(t) \frac{\mathcal{J}(u + tw) - \mathcal{J}(u)}{t} \geq \operatorname{sgn}(t) \lambda \int_{\Omega} f(x, u) w dx.$$

Letting $t \rightarrow 0$ we get, as \mathcal{J} restricted to $W_0^{1,1}(\Omega)$ is Gateaux differentiable,

$$\pm \mathcal{J}'(u)w \geq \pm \lambda \int_{\Omega} f(x, u)w \, dx,$$

that is

$$\mathcal{J}'(u)w = \lambda \int_{\Omega} f(x, u)w \, dx.$$

Remark 2.4. A strong solution u of (1) is a weak solution of (1). Indeed, we have $f(\cdot, u) \in L^p(\Omega)$ and, by the Dirichlet boundary condition, $u \in W_0^{1,1}(\Omega)$. Multiplying the equation in (9) by $v \in W_0^{1,1}(\Omega)$, integrating on Ω and using the Green's formula, we obtain (8).

Conversely, a weak solution u of (1), with $u \in W^{2,p}(\Omega)$ for some $p > N$, is a strong solution of (1). Indeed, using the Green's formula, we see that u satisfies the equation in (9) a.e. in Ω . The Dirichlet boundary condition is satisfied as well since $u \in W_0^{1,1}(\Omega) \cap C^0(\bar{\Omega})$.

Lower and upper solutions (see [20, Section 12], [27] and [36]). Assume (h_1) . A function $\alpha \in BV(\Omega)$ is said to be a lower solution of (1), for a given $\lambda > 0$, if $f(\cdot, \alpha) \in L^p(\Omega)$ for some $p > N$ and

$$\mathcal{J}(\alpha + z) - \mathcal{J}(\alpha) \geq \lambda \int_{\Omega} f(x, \alpha)z \, dx \quad (10)$$

for every $z \in BV(\Omega)$ with $z(x) \leq 0$ a.e. in Ω . Similarly a function $\beta \in BV(\Omega)$ is said to be an upper solution of (1), for a given $\lambda > 0$, if $f(\cdot, \beta) \in L^p(\Omega)$ for some $p > N$ and

$$\mathcal{J}(\beta + z) - \mathcal{J}(\beta) \geq \lambda \int_{\Omega} f(x, \beta)z \, dx$$

for every $z \in BV(\Omega)$ with $z(x) \geq 0$ a.e. in Ω . It follows from [36, Remark 2.3] that u is a solution of (1) if and only if it is simultaneously a lower and an upper solution of (1).

Remark 2.5. Note that α is a lower solution of (1) if and only if it is a minimizer of the functional

$$\mathcal{H}_{\lambda, \alpha}(v) = \mathcal{J}(v) - \lambda \mathcal{F}'(\alpha)(v)$$

in the cone $C_{\alpha} = \{v \in BV(\Omega) \mid v(x) \leq \alpha(x) \text{ a.e. in } \Omega\}$. Similarly β is an upper solution of (1) if and only if it is a minimizer of the functional

$$\mathcal{H}_{\lambda, \beta}(v) = \mathcal{J}(v) - \lambda \mathcal{F}'(\beta)(v)$$

in the cone $C_{\beta} = \{v \in BV(\Omega) \mid v(x) \geq \beta(x) \text{ a.e. in } \Omega\}$.

Remark 2.6. Assume (h_1) . Suppose that $\alpha \in W^{1,1}(\Omega)$ is such that $f(\cdot, \alpha) \in L^p(\Omega)$ for some $p > N$, $\alpha|_{\partial\Omega}(x) \leq 0$ for \mathcal{H}_{N-1} -a.e. $x \in \partial\Omega$ and

$$\int_{\Omega} \nabla \alpha \nabla z / \sqrt{1 + |\nabla \alpha|^2} \, dx + \int_{\partial\Omega} |z|_{\partial\Omega} \, d\mathcal{H}_{N-1} \geq \lambda \int_{\Omega} f(x, \alpha)z \, dx$$

for every $z \in W^{1,1}(\Omega)$, with $z(x) \leq 0$ a.e. in Ω . Then (see [36, Lemma 3.8]) α is a lower solution of (1). A similar observation can be made about an upper solution β (see [36, Lemma 3.7]).

From [36, Theorem 2.4] we derive the following result (see also [27, Theorem 3.2] for a related statement).

Proposition 2.1. Assume (h_1) and (h_2) . Suppose that there exist a lower solution α and an upper solution β of (1), for a given $\lambda > 0$, such that $\alpha(x) \leq \beta(x)$ a.e. in Ω and $F(\cdot, \alpha) \in L^1(\Omega)$, or $F(\cdot, \beta) \in L^1(\Omega)$. Assume further that there are $p > N$ and $\gamma \in L^p(\Omega)$ such that $|f(x, s)| \leq \gamma(x)$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, with $\alpha(x) \leq s \leq \beta(x)$. Then problem (1) has at least one solution u such that

$$\alpha(x) \leq u(x) \leq \beta(x) \quad \text{a.e. in } \Omega$$

and

$$\mathcal{I}_\lambda(u) = \min \{ \mathcal{I}_\lambda(v) \mid v \in BV(\Omega), \alpha(x) \leq v(x) \leq \beta(x) \text{ a.e. in } \Omega \}.$$

Spectral constants. We denote by

$$\lambda_1 = \min_{H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} \quad (11)$$

the principal eigenvalue of $-\Delta$ with Dirichlet boundary conditions. Assuming (h_1) , we denote by

$$\mu_1 = \min_{BV(\Omega) \setminus \{0\}} \frac{\int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}_{N-1}}{\int_\Omega |u| dx} \quad (12)$$

the principal eigenvalue of $-\Delta_1$ with Dirichlet boundary conditions (see [13]). Note that μ_1 is the best Poincaré constant appearing in (4) when $p = 1$.

Denote by S^{N-1} the unit sphere in \mathbb{R}^N . For each $e \in S^{N-1}$ we set

$$a_e(\Omega) = \inf_{x \in \Omega} xe, \quad b_e(\Omega) = \sup_{x \in \Omega} xe, \quad L_e(\Omega) = b_e(\Omega) - a_e(\Omega). \quad (13)$$

Note that $L_e(\Omega)$ continuously depends on $e \in S^{N-1}$ and $\inf_{e \in S^{N-1}} L_e(\Omega) = \min_{e \in S^{N-1}} L_e(\Omega) > 0$. Define $L(\Omega) = \min_{e \in S^{N-1}} L_e(\Omega)$. We then set

$$\lambda_1^* = \left(\frac{\pi}{L(\Omega)} \right)^2 \quad (14)$$

and

$$\mu_1^* = \frac{2}{L(\Omega)}. \quad (15)$$

Note that λ_1^* and μ_1^* are, respectively, the principal eigenvalues of $-\Delta$ and $-\Delta_1$, with Dirichlet boundary conditions, in the interval $[0, L(\Omega)]$.

Denote by $R(\Omega) > 0$ the largest $R > 0$ such that there is an open ball of radius R contained in Ω . We set

$$\lambda_1^\sharp = (2^N - 1) \left(\frac{2}{R(\Omega)} \right)^2. \quad (16)$$

Let $C (\subseteq \Omega)$ be a Caccioppoli set (see [20, p. 6]) and let χ_C be its characteristic function. Denote by

$$\text{Per}(C) = \int_{\Omega} |D\chi_C| + \int_{\partial\Omega} \chi_C|_{\partial\Omega} d\mathcal{H}_{N-1} \quad (17)$$

the perimeter of C in \mathbb{R}^N . We set

$$\mu_1^\# = \frac{\text{Per}(\Omega)}{\text{meas}(\Omega)}. \quad (18)$$

Note that $\lambda_1^*, \lambda_1 \leq \lambda_1^\#$ and $\mu_1^*, \mu_1 \leq \mu_1^\#$ and, for some special geometries of Ω , $\mu_1 = \mu_1^\#$ (see e.g. [13, Proposition 11]).

An elementary inequality. For any given $\eta \in]0, 1[$ there exists $d > 0$ such that, for every $s \in \mathbb{R}$,

$$\frac{1}{2}(1-\eta)|s| - d \leq \sqrt{1+s^2} - 1 - \eta \frac{s^2}{\sqrt{1+s^2}} \leq (1-\eta)|s|. \quad (19)$$

3. Existence and multiplicity results

In this section we prove several existence and multiplicity results for problem (1), assuming various conditions on the behavior at 0 or at $+\infty$ of the potential F of f , so as to extend the model statements presented in the Introduction.

3.1. Existence of at least one positive solution

3.1.1. Potential subquadratic at zero

The relevant assumption in this context is (h_7) , which expresses a form of local and desultory subquadraticity of the potential F at 0.

Theorem 3.1. *Assume*

- (h_4) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{1,\sigma}$ boundary $\partial\Omega$ for some $\sigma \in]0, 1[$;
- (h_2) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions;
- (h_5) $f(x, 0) \geq 0$ for a.e. $x \in \Omega$;
- (h_6) there exist constants $r > 0$ and $c > 0$ such that $|f(x, s)| \leq c$ for a.e. $x \in \Omega$ and every $s \in [0, r]$;
- (h_7) there exist open sets ω and ω_1 , with $\bar{\omega} \subset \omega_1 \subseteq \Omega$, such that

$$\limsup_{s \rightarrow 0^+} \frac{\int_{\omega} F(x, s) dx}{s^2} = +\infty$$

and

$$\liminf_{s \rightarrow 0^+} \frac{\int_{\omega_1 \setminus \omega} F(x, s) dx}{s^2} > -\infty.$$

Then there exists $\lambda^* \in]0, +\infty]$ such that, for every $\lambda \in]0, \lambda^*[$, problem (1) has at least one positive weak solution $u_\lambda \in C^{1,\tau}(\bar{\Omega})$, for some $\tau \in]0, 1[$, satisfying

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_{C^1(\bar{\Omega})} = 0 \quad \text{and} \quad \mathcal{I}_\lambda(u_\lambda) < \mathcal{I}_\lambda(0) = \text{meas}(\Omega).$$

Proof. *Step 1. A modified problem.* Let $a : [0, +\infty[\rightarrow [0, +\infty[$ be the $C^{1,1}$ non-increasing function defined by

$$a(s) = \begin{cases} (1+s)^{-1/2} & \text{if } s \in [0, 1[, \\ \frac{\sqrt{2}}{16}(s-2)^2 + \frac{7\sqrt{2}}{16} & \text{if } s \in [1, 2[, \\ \frac{7\sqrt{2}}{16} & \text{if } s \in [2, +\infty[. \end{cases} \quad (20)$$

Set, for every $s \geq 0$,

$$A(s) = \int_0^s a(t) dt. \quad (21)$$

Note that the structure and the regularity conditions assumed in [18] are satisfied. Further, we have for every $s \geq 0$

$$\frac{7\sqrt{2}}{16} \leq a(s) \leq 1 \quad (22)$$

and hence

$$\frac{7\sqrt{2}}{16}s \leq A(s) \leq s. \quad (23)$$

Let $\chi : [0, +\infty[\rightarrow [0, 1]$ be a continuous function such that

$$\chi(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \frac{r}{2}, \\ 0 & \text{if } s \geq r, \end{cases}$$

where r is defined in (h_6) . Then we set, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$,

$$g(x, s) = \begin{cases} \chi(-s)f(x, 0) & \text{if } s < 0, \\ \chi(s)f(x, s) & \text{if } s \geq 0, \end{cases} \quad (24)$$

and

$$G(x, s) = \int_0^s g(x, t) dt.$$

Note that, by (h_6) ,

$$|g(x, s)| \leq c \quad (25)$$

and

$$|G(x, s)| \leq cr \quad (26)$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. Let us consider the modified problem

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^2)\nabla u) = \lambda g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (27)$$

A solution of (27) is a function $u \in H_0^1(\Omega)$ satisfying

$$\int_{\Omega} a(|\nabla u|^2) \nabla u \nabla v \, dx = \lambda \int_{\Omega} g(x, u) v \, dx \quad (28)$$

for every $v \in H_0^1(\Omega)$. For each $\lambda > 0$ we define the functional $\mathcal{K}_{\lambda} : H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$\mathcal{K}_{\lambda}(u) = \frac{1}{2} \int_{\Omega} A(|\nabla u|^2) \, dx - \lambda \int_{\Omega} G(x, u) \, dx. \quad (29)$$

\mathcal{K}_{λ} is of class C^1 and weakly lower semicontinuous, being the sum of a convex and a weakly continuous function. Moreover, $u \in H_0^1(\Omega)$ is a solution of (27) if and only if u is a critical point of \mathcal{K}_{λ} .

Step 2. Existence of solutions of the modified problem for every $\lambda > 0$. Fix any $\lambda > 0$. By (23) and (26) the functional \mathcal{K}_{λ} is coercive and bounded from below in $H_0^1(\Omega)$; hence it has a global minimizer $u_{\lambda} \in H_0^1(\Omega)$. Take $w \in H_0^1(\Omega)$ such that $w(x) \geq 0$ in Ω , $w(x) = 0$ in $\Omega \setminus \omega_1$ and $w(x) = 1$ in ω , ω and ω_1 being defined in (h_7) . By (h_7) there exist a sequence $(d_n)_n$, with $d_n > 0$ for every n and $\lim_{n \rightarrow +\infty} d_n = 0$, and a constant $\kappa_1 > 0$, such that

$$\lim_{n \rightarrow +\infty} d_n^{-2} \int_{\omega} G(x, d_n) \, dx = +\infty \quad (30)$$

and

$$\int_{\omega_1 \setminus \omega} G(x, d_n w) \, dx \geq -\kappa_1 d_n^2 \int_{\omega_1 \setminus \omega} w^2 \, dx. \quad (31)$$

Hence, we have

$$\begin{aligned} \mathcal{K}_{\lambda}(d_n w) &= \frac{1}{2} \int_{\Omega} A(d_n^2 |\nabla w|^2) \, dx - \lambda \int_{\omega} G(x, d_n) \, dx - \lambda \int_{\Omega \setminus \omega} G(x, d_n w) \, dx \\ &\leq d_n^2 \left(\frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dx - \lambda d_n^{-2} \int_{\omega} G(x, d_n) \, dx - \lambda \kappa_1 \int_{\omega_1 \setminus \omega} w^2 \, dx \right) < 0, \end{aligned}$$

for all n large enough. This implies that

$$\mathcal{K}_{\lambda}(u_{\lambda}) = \min_{u \in H_0^1(\Omega)} \mathcal{K}_{\lambda}(u) < 0 \quad (32)$$

and hence $u_{\lambda} \neq 0$. Testing (28) against $(u_{\lambda} - r)^+$, which belongs to $H_0^1(\Omega)$ by Stampacchia theorem (see [44, Section 1.8]), and using (22) and (24), we get

$$\begin{aligned} \frac{7\sqrt{2}}{16} \int_{\Omega} |\nabla(u_{\lambda} - r)^+|^2 \, dx &\leq \int_{\Omega} a(|\nabla u_{\lambda}|^2) |\nabla(u_{\lambda} - r)^+|^2 \, dx = \int_{\Omega} a(|\nabla u_{\lambda}|^2) \nabla u_{\lambda} \nabla(u_{\lambda} - r)^+ \, dx \\ &= \lambda \int_{\Omega} \chi(u_{\lambda}) f(x, u_{\lambda})(u_{\lambda} - r)^+ \, dx = 0. \end{aligned}$$

Therefore we have $(u_\lambda - r)^+ = 0$, i.e. $u_\lambda(x) \leq r$ a.e. in Ω . Testing against $-u_\lambda^-$ and using (22), (24) and (h_5) , we obtain

$$\begin{aligned} \frac{7\sqrt{2}}{16} \int_{\Omega} |\nabla u_\lambda^-|^2 dx &\leq \int_{\Omega} a(|\nabla u_\lambda^-|^2) |\nabla u_\lambda^-|^2 dx \\ &= - \int_{\Omega} a(|\nabla u_\lambda|^2) \nabla u_\lambda \nabla u_\lambda^- dx \\ &= -\lambda \int_{\Omega} g(x, u_\lambda) u_\lambda^- dx = -\lambda \int_{\Omega} \chi(u_\lambda^-) f(x, 0) u_\lambda^- dx \leq 0. \end{aligned}$$

Therefore we have $u_\lambda^- = 0$, i.e. $u_\lambda(x) \geq 0$ a.e. in Ω . Thus we conclude that for a.e. $x \in \Omega$

$$0 \leq u_\lambda(x) \leq r. \quad (33)$$

Due to (33) and (25), the regularity theory for (27) (see [18]) implies that there exist $\tau \in]0, 1]$ and $\kappa_2 > 0$ such that

$$\|u_\lambda\|_{C^{1,\tau}(\bar{\Omega})} \leq \kappa_2 \quad (34)$$

for every $\lambda \in]0, 1]$.

Step 3. There exists $\lambda^ \in]0, +\infty]$ such that, for every $\lambda \in]0, \lambda^*[$, problem (1) has at least one positive weak solution $u_\lambda \in C^{1,\tau}(\bar{\Omega})$, for some $\tau \in]0, 1]$, satisfying*

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_{C^1(\bar{\Omega})} = 0.$$

Pick any sequence $(\lambda_n)_n$, with $\lambda_n \in]0, 1]$ and $\lim_{n \rightarrow +\infty} \lambda_n = 0$, and let $(u_{\lambda_n})_n$ be the corresponding sequence of solutions of (27) we have found in Step 2. Estimate (34) and the Arzelà–Ascoli theorem yield the existence of a subsequence $(u_k)_k = (u_{\lambda_{n_k}})_k$ converging in $C^1(\bar{\Omega})$ to some function $u \in C^1(\bar{\Omega})$ with $u(x) = 0$ on $\partial\Omega$. Testing (28) against u_k and using (22), (25) and (33), we get

$$\begin{aligned} \frac{7\sqrt{2}}{16} \int_{\Omega} |\nabla u_k|^2 dx &\leq \int_{\Omega} a(|\nabla u_k|^2) |\nabla u_k|^2 dx \\ &= \lambda_{n_k} \int_{\Omega} g(x, u_k) u_k dx \leq \lambda_{n_k} c r \text{meas}(\Omega) \end{aligned}$$

and hence, passing to the limit, $u = 0$. Therefore we conclude that

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_{C^1(\bar{\Omega})} = 0.$$

This implies that there exists $\lambda^* \in]0, +\infty]$ such that, for every $\lambda \in]0, \lambda^*[$, $u_\lambda \in H_0^1(\Omega) \cap C^{1,\tau}(\bar{\Omega})$ is a positive weak solution of (1). Since $\mathcal{K}_\lambda(v) = \mathcal{I}_\lambda(v) - \text{meas}(\Omega)$ for any $v \in C^1(\bar{\Omega}) \cap H_0^1(\Omega)$ with $\|u\|_{C^1(\bar{\Omega})} < \min\{1, \frac{r}{2}\}$, by (32) we also conclude $\mathcal{I}_\lambda(u_\lambda) < \mathcal{I}_\lambda(0) = \text{meas}(\Omega)$. \square

Remark 3.1. Assumptions (h_5) and (h_7) are implied by the following subquadraticity condition at 0.

$$(h_8) \quad \liminf_{s \rightarrow 0^+} \frac{F(x, s)}{s^2} = +\infty \text{ uniformly a.e. in } \Omega.$$

Remark 3.2. If, in addition to all assumptions of Theorem 3.1, we suppose that

$$(h_9) \quad \text{there exists a constant } r > 0 \text{ such that } f(x, s) \geq 0 \text{ for a.e. } x \in \Omega \text{ and every } s \in [0, r],$$

then the strong maximum principle and the boundary point lemma [41, Corollary 8.3, Corollary 8.4] yield $u_\lambda(x) > 0$ for every $x \in \Omega$ and $\frac{\partial u_\lambda}{\partial \nu}(x) < 0$ for every $x \in \partial\Omega$, ν being the unit outer normal to Ω at $x \in \partial\Omega$. Note that, under (h_9) , the second condition in (h_7) is automatically satisfied.

3.1.2. Potential sublinear at infinity

The relevant assumptions in this context are (h_{11}) in Theorem 3.2 and (h_{14}) in Theorem 3.3. Condition (h_{11}) requires F to be sublinear at $+\infty$; whereas condition (h_{14}) allows F to be just desultorily sublinear at $+\infty$. The two hypotheses are however independent, because (h_{14}) , although weaker than (h_{11}) when f is autonomous, requires otherwise an additional uniform control on f .

Theorem 3.2. Assume

(h_1) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{0,1}$ boundary $\partial\Omega$;

(h_2) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions;

(h_5) $f(x, 0) \geq 0$ for a.e. $x \in \Omega$;

(h_{10}) there exist constants $q \in]1, \frac{N}{N-1}[$, $c_1 > 0$ and a function $c_2 \in L^{\frac{q}{q-1}}(\Omega)$ such that

$$|f(x, s)| \leq c_1 s^{q-1} + c_2(x)$$

for a.e. $x \in \Omega$ and every $s \in [0, +\infty[$;

(h_{11}) $\limsup_{s \rightarrow +\infty} \frac{F(x, s)}{s} \leq 0$ uniformly a.e. in Ω ;

(h_{12}) there exists $s_0 > 0$ such that $\int_\Omega F(x, s_0) dx > 0$.

Then there exists $\lambda_* \in [0, +\infty[$ such that, for every $\lambda \in]\lambda_*, +\infty[$, problem (1) has at least one positive solution u_λ , satisfying

$$\lim_{\lambda \rightarrow +\infty} \mathcal{I}_\lambda(u_\lambda) = -\infty \quad \text{and} \quad \liminf_{\lambda \rightarrow +\infty} \|u_\lambda\|_{L^q(\Omega)} > 0.$$

Proof. Since we are looking for positive solutions of (1), we can modify f by setting

$$f(x, s) = f(x, 0) - \arctan(s) \tag{35}$$

for a.e. $x \in \Omega$ and every $s < 0$. We derive from (h_{10}) that, with possibly a different choice of the function c_2 ,

$$|f(x, s)| \leq c_1 |s|^{q-1} + c_2(x) \tag{36}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. Moreover, from (h_5) , (35) and (h_{11}) we deduce that, for every $\varepsilon > 0$, there exists $s_\varepsilon > 0$ such that

$$F(x, s) \leq \varepsilon |s| \tag{37}$$

for a.e. $x \in \Omega$ and every $|s| \geq s_\varepsilon$.

Step 1. For each $\lambda > 0$ there exists $\min_{v \in BV(\Omega)} \mathcal{I}_\lambda(v)$. Fix $\lambda > 0$ and pick $\varepsilon > 0$ such that $\varepsilon < \frac{\mu_1}{\lambda}$, where μ_1 is defined by (12). By (36) and (37) there exists $c_3 \in L^{\frac{q}{q-1}}(\Omega)$ such that, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$,

$$F(x, s) \leq \varepsilon |s| + c_3(x).$$

Using the Poincaré inequality we get, for every $v \in BV(\Omega)$,

$$\mathcal{I}_\lambda(v) \geq \int_{\Omega} |Dv| + \int_{\partial\Omega} |v|_{\partial\Omega} d\mathcal{H}_{N-1} - \lambda \varepsilon \int_{\Omega} |v| dx - c_4 \geq \left(1 - \varepsilon \frac{\lambda}{\mu_1}\right) \|v\|_{BV(\Omega)} - c_4,$$

for some constant $c_4 > 0$. Therefore \mathcal{I}_λ is bounded from below and coercive in $BV(\Omega)$. Let $(u_n)_n$ be a minimizing sequence. Since $(u_n)_n$ is bounded in $BV(\Omega)$, by the compact embedding of $BV(\Omega)$ into $L^q(\Omega)$, there exist a subsequence of $(u_n)_n$, which we still denote by $(u_n)_n$, and a function $u_\lambda \in BV(\Omega)$ such that $\lim_{n \rightarrow +\infty} u_n = u_\lambda$ in $L^q(\Omega)$. As \mathcal{I}_λ is lower semicontinuous with respect to the L^q -convergence in $BV(\Omega)$, we have

$$\liminf_{n \rightarrow +\infty} \mathcal{I}_\lambda(u_n) \geq \mathcal{I}_\lambda(u_\lambda).$$

Hence we conclude that

$$\inf_{v \in BV(\Omega)} \mathcal{I}_\lambda(v) = \lim_{n \rightarrow +\infty} \mathcal{I}_\lambda(u_n) \geq \mathcal{I}_\lambda(u_\lambda),$$

that is

$$\mathcal{I}_\lambda(u_\lambda) = \min_{v \in BV(\Omega)} \mathcal{I}_\lambda(v). \quad (38)$$

Step 2. For each $\lambda > 0$ there exists a solution u_λ of (1) with $u_\lambda(x) \geq 0$ a.e. in Ω . Fix $\lambda > 0$. Any minimizer u_λ of \mathcal{I}_λ , whose existence follows from Step 1, is a solution of (1). Let us prove that $u_\lambda = u_\lambda^+$. Since, by (h_5) and (h_{10}) , 0 is a lower solution of (1), using $-u_\lambda^-$ as a test function in (10) we get

$$\mathcal{J}(-u_\lambda^-) - \mathcal{J}(0) \geq -\lambda \int_{\Omega} f(x, 0) u_\lambda^- dx. \quad (39)$$

Moreover, as u_λ is a solution of (1), using u_λ^+ as a test function in (2), we have

$$\begin{aligned} \mathcal{J}(u_\lambda^+) - \mathcal{J}(u_\lambda) &\geq \lambda \int_{\Omega} f(x, u_\lambda) (u_\lambda^+ - u_\lambda) dx \\ &= \lambda \int_{\Omega} f(x, 0) u_\lambda^- dx - \lambda \int_{\Omega} \arctan(-u_\lambda^-) u_\lambda^- dx. \end{aligned} \quad (40)$$

Summing up (39) and (40) and using (5), with $u = u_\lambda$ and $v = 0$, we obtain

$$0 \geq \mathcal{J}(u_\lambda^+) + \mathcal{J}(-u_\lambda^-) - \mathcal{J}(u_\lambda) - \mathcal{J}(0) \geq \lambda \int_{\Omega} \arctan(u_\lambda^-) u_\lambda^- dx \geq 0.$$

This yields $u_\lambda^- = 0$.

Step 3. There is $\lambda_* \geq 0$ such that, for each $\lambda > \lambda_*$, there exists a positive solution u_λ of (1) satisfying

$$\lim_{\lambda \rightarrow +\infty} \mathcal{I}_\lambda(u_\lambda) = -\infty \quad \text{and} \quad \liminf_{\lambda \rightarrow +\infty} \|u_\lambda\|_{L^q(\Omega)} > 0.$$

For any $\lambda > 0$ let u_λ be a minimizer of \mathcal{I}_λ , whose existence follows from Step 1. From (38) and (h_{12}) we can find $\lambda_* \geq 0$ such that, for every $\lambda > \lambda_*$, we have

$$\mathcal{I}_\lambda(u_\lambda) \leq \mathcal{I}_\lambda(s_0) = \mathcal{I}_\lambda(0) + s_0 \operatorname{Per}(\Omega) - \lambda \int_{\Omega} F(x, s_0) dx < \mathcal{I}_\lambda(0).$$

Hence we infer that $u_\lambda \neq 0$ and, by Step 2, it is a positive solution of (1). Moreover, letting $\lambda \rightarrow +\infty$, we also get

$$\lim_{\lambda \rightarrow +\infty} \mathcal{I}_\lambda(u_\lambda) = -\infty.$$

Take $\lambda_2 > \lambda_1 > \lambda_*$. We want to prove that $\mathcal{F}(u_{\lambda_1}) \leq \mathcal{F}(u_{\lambda_2})$. Indeed, otherwise we should get

$$\begin{aligned} \mathcal{J}(u_{\lambda_2}) - \lambda_2 \mathcal{F}(u_{\lambda_2}) &= \mathcal{J}(u_{\lambda_2}) - \lambda_1 \mathcal{F}(u_{\lambda_2}) - (\lambda_2 - \lambda_1) \mathcal{F}(u_{\lambda_2}) \\ &\geq \mathcal{J}(u_{\lambda_1}) - \lambda_1 \mathcal{F}(u_{\lambda_1}) - (\lambda_2 - \lambda_1) \mathcal{F}(u_{\lambda_2}) \\ &> \mathcal{J}(u_{\lambda_1}) - \lambda_1 \mathcal{F}(u_{\lambda_1}) - (\lambda_2 - \lambda_1) \mathcal{F}(u_{\lambda_1}) \\ &= \mathcal{J}(u_{\lambda_1}) - \lambda_2 \mathcal{F}(u_{\lambda_1}) \geq \mathcal{J}(u_{\lambda_2}) - \lambda_2 \mathcal{F}(u_{\lambda_2}), \end{aligned}$$

which is a contradiction. Moreover, as for each $\lambda > \lambda_*$ we have $\mathcal{I}_\lambda(u_\lambda) < \mathcal{I}_\lambda(0) = \operatorname{meas}(\Omega)$ and $\mathcal{J}(u_\lambda) \geq \operatorname{meas}(\Omega)$, we infer $\mathcal{F}(u_\lambda) > 0$. Assume now, by contradiction, that there exists an increasing sequence $(\lambda_n)_n$, with $\lambda_n \geq \lambda_*$ for every n and $\lim_{n \rightarrow +\infty} \lambda_n = +\infty$, such that $\lim_{n \rightarrow +\infty} u_{\lambda_n} = 0$ in $L^q(\Omega)$. The continuity of $\mathcal{F} : L^q(\Omega) \rightarrow \mathbb{R}$ yields $\lim_{n \rightarrow +\infty} \mathcal{F}(u_{\lambda_n}) = \mathcal{F}(0) = 0$, thus contradicting the fact that $\mathcal{F}(u_{\lambda_n}) \geq \mathcal{F}(u_{\lambda_1}) > 0$ for every $n \geq 1$. Hence we conclude that

$$\liminf_{\lambda \rightarrow +\infty} \|u_\lambda\|_{L^q(\Omega)} > 0. \quad \square$$

Theorem 3.3. Assume

- (h_1) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{0,1}$ boundary $\partial\Omega$;
- (h_{13}) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the L^p -Carathéodory conditions for some $p > N$, i.e. f is a Carathéodory function and, for each $r > 0$, there exists $\gamma_r \in L^p(\Omega)$ such that $|f(x, s)| \leq \gamma_r(x)$ for a.e. $x \in \Omega$ and every $s \in [-r, r]$;
- (h_5) $f(x, 0) \geq 0$ for a.e. $x \in \Omega$;
- (h_{12}) there exists $s_0 > 0$ such that $\int_{\Omega} F(x, s_0) dx > 0$;
- (h_{14}) there exist a constant r and a continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x, s) \leq h(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \geq r$$

and

$$\liminf_{s \rightarrow +\infty} \frac{H(s)}{s} \leq 0,$$

where $H(s) = \int_0^s h(t) dt$.

Then there exists $\lambda_* \in [0, +\infty[$ such that, for every $\lambda \in]\lambda_*, +\infty[$, problem (1) has at least one positive solution u_λ , with $u_\lambda \in L^\infty(\Omega)$, satisfying

$$\lim_{\lambda \rightarrow +\infty} \mathcal{I}_\lambda(u_\lambda) = -\infty.$$

Proof. We start proving the following result, which is related to [36, Lemma 3.9].

Claim. Assume (h_1) . Suppose that $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the L^p -Carathéodory conditions for some $p > N$ and there exist a constant $r > 0$ and a continuous function $\ell : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$g(x, s) \leq \ell(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \geq r$$

and

$$\liminf_{s \rightarrow +\infty} \frac{\mathcal{L}(s)}{s} < \mu_1^*,$$

where $\mathcal{L}(s) = \int_0^s \ell(t) dt$ and μ_1^* is defined by (15). Then there exists a sequence $(\beta_n)_n$ of upper solutions of the problem

$$\begin{cases} -\operatorname{div}(\nabla u / \sqrt{1 + |\nabla u|^2}) = g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (41)$$

such that, for each n , $\beta_n \in C^2(\bar{\Omega})$ and $\lim_{n \rightarrow +\infty} (\min_{\bar{\Omega}} \beta_n) = +\infty$.

Suppose first that $\sup\{s > 0 \mid \ell(s) \leq 0\} = +\infty$. Then there exists an increasing sequence $(\beta_n)_n$ of constant upper solutions of (41) with $\lim_{n \rightarrow +\infty} \beta_n = +\infty$. Therefore we may assume $\ell(s) > 0$ in $[r, +\infty[$. Possibly replacing $\ell(s)$ with $\ell(r)$ in $] -\infty, r[$, we can further suppose that $\ell(s) > 0$ in \mathbb{R} . Fix $\mu > 0$ such that

$$\liminf_{s \rightarrow +\infty} \frac{\mathcal{L}(s)}{s} < \mu \leq \mu_1^*.$$

Then we can find an increasing sequence $(R_n)_n$ such that $\lim_{n \rightarrow +\infty} R_n = +\infty$ and, for every n ,

$$r + \frac{1}{\mu} < R_n < R_{n+1} - \frac{1}{\mu}$$

and

$$\mathcal{L}(R_n) - \mathcal{L}(s) < \mu(R_n - s) \quad \text{in } \left[R_n - \frac{1}{\mu}, R_n \right]. \quad (42)$$

Fix n , set $R = R_n$ and consider the initial value problem

$$-(v' / \sqrt{1 + |v'|^2})' = \ell(v), \quad v(0) = R, \quad v'(0) = 0. \quad (43)$$

Let $v \in C^2(-\omega, \omega[)$ be an even non-extendible solution of (43). Then $v|_{[0, \omega[}$ is decreasing, concave and satisfies the energy relation

$$1 - \frac{1}{\sqrt{1 + |v'(t)|^2}} = \mathcal{L}(R) - \mathcal{L}(v(t)) \quad (44)$$

in $[0, \omega[$. Define

$$T = \sup \left\{ t \in [0, \omega[\mid v(t) > R - \frac{1}{\mu} \right\}.$$

Since $v|_{[0, \omega[}$ is decreasing and concave we have $T < +\infty$. We set

$$v(T) = \lim_{t \rightarrow T} v(t) \geq R - \frac{1}{\mu}.$$

Note that $v \in C^2([-T, T]) \cap C^0([-T, T])$ and it may be $v'(T) = -\infty$. Using (44), (42) and the fact that the function $t \mapsto (1-t)/\sqrt{2t-t^2}$ is decreasing in $]0, 1]$, we get

$$\begin{aligned} T &= \int_0^T -v'(t) \frac{1 - (\mathcal{L}(R) - \mathcal{L}(v(t)))}{\sqrt{2(\mathcal{L}(R) - \mathcal{L}(v(t))) - (\mathcal{L}(R) - \mathcal{L}(v(t)))^2}} dt \\ &= \int_{v(T)}^{v(0)} \frac{1 - (\mathcal{L}(R) - \mathcal{L}(s))}{\sqrt{2(\mathcal{L}(R) - \mathcal{L}(s)) - (\mathcal{L}(R) - \mathcal{L}(s))^2}} ds \\ &\geq \int_{R - \frac{1}{\mu}}^R \frac{1 - \mu(R-s)}{\sqrt{2\mu(R-s) - (\mu(R-s))^2}} ds = \frac{1}{\mu} \int_0^1 \frac{1-t}{\sqrt{2t-t^2}} dt = \frac{1}{\mu} \geq \frac{1}{\mu_1^*}. \end{aligned}$$

Let $\hat{e} \in S^{N-1}$ be such that $L_{\hat{e}}(\Omega) = \min_{e \in S^{N-1}} L_e(\Omega)$ and set, for every $x \in \bar{\Omega}$,

$$\beta(x) = v \left(x\hat{e} - \frac{1}{2}(a_{\hat{e}}(\Omega) + b_{\hat{e}}(\Omega)) \right),$$

where $L_{\hat{e}}(\Omega)$, $a_{\hat{e}}(\Omega)$ and $b_{\hat{e}}(\Omega)$ are defined in (13). Observe that $\beta \in C^2(\Omega) \cap C^0(\bar{\Omega})$, $R - \frac{1}{\mu} \leq \beta(x) \leq R$ for every $x \in \Omega$. As

$$\int_{-T}^T |v'| dt = 2(v(0) - v(T)),$$

we see that $v \in W^{1,1}(-T, T)$ and hence $\beta \in W^{1,1}(\Omega)$. Moreover we have

$$-\operatorname{div}(\nabla \beta / \sqrt{1 + |\nabla \beta|^2}) = -v'' / (1 + |v'|^2)^{\frac{3}{2}} = \ell(v) \geq g(x, \beta) \quad (45)$$

a.e. in Ω . Note in particular that $\nabla \beta / \sqrt{1 + |\nabla \beta|^2} \in C^1(\bar{\Omega})$ and $g(\cdot, \beta) \in L^p(\Omega)$. Take $z \in W^{1,1}(\Omega)$ such that $z(x) \geq 0$ a.e. in Ω . Multiplying (45) by z and integrating, we get by the Green's formula

$$\int_{\Omega} \nabla \beta \nabla z / \sqrt{1 + |\nabla \beta|^2} dx + \int_{\partial \Omega} |z|_{\partial \Omega} d\mathcal{H}_{N-1} \geq \int_{\Omega} g(x, \beta) z dx.$$

By Remark 2.6, β is an upper solution of (1). We finally stress that, if $\mu < \mu_1^*$, then $\beta \in C^2(\bar{\Omega})$. This concludes the proof of the claim.

Step 1. For each $\lambda > 0$ there exists a solution u_λ of (1) with $u_\lambda(x) \geq 0$ a.e. in Ω . Fix $\lambda > 0$. Conditions (h_5) and (h_{13}) imply that $\alpha = 0$ is a lower solution of (1). Let us set $\lambda f = g$ and $\lambda h = \ell$. Conditions (h_{13}) and (h_{14}) imply that g and ℓ satisfy the assumptions of the claim. Hence there exists an upper solution $\beta_\lambda \in C^2(\bar{\Omega})$ of (1) such that

$$\min_{\bar{\Omega}} \beta_\lambda > s_0, \quad (46)$$

where s_0 is given in (h_{12}) . By Proposition 2.1 there exists a solution u_λ of (1) such that $0 \leq u_\lambda(x) \leq \beta_\lambda(x)$ a.e. in Ω and

$$\mathcal{I}_\lambda(u_\lambda) = \min\{\mathcal{I}_\lambda(v) \mid v \in BV(\Omega), 0 \leq v(x) \leq \beta_\lambda(x) \text{ a.e. in } \Omega\}. \quad (47)$$

Step 2. There is $\lambda_* \geq 0$ such that, for each $\lambda > \lambda_*$, there exists a positive solution u_λ of (1) satisfying

$$\lim_{\lambda \rightarrow +\infty} \mathcal{I}_\lambda(u_\lambda) = -\infty.$$

From (46) and (47) we infer that

$$\mathcal{I}_\lambda(u_\lambda) \leq \mathcal{I}_\lambda(s_0) = \mathcal{I}_\lambda(0) + s_0 \operatorname{Per}(\Omega) - \lambda \int_{\Omega} F(x, s_0) dx.$$

By (h_{12}) , letting $\lambda \rightarrow +\infty$, we get the conclusion. \square

3.1.3. Potential subquadratic at zero and sublinear at infinity

In the following theorem we show that the existence of positive solutions of (1) for any given $\lambda > 0$ can be established if the potential F is desultorily subquadratic at 0 and sublinear at $+\infty$. Loosely speaking the condition at $+\infty$ yields the existence of a solution and the conditions at 0 guarantee that it is positive.

Theorem 3.4. Assume

- (h_1) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{0,1}$ boundary $\partial\Omega$;
- (h_2) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions;
- (h_5) $f(x, 0) \geq 0$ for a.e. $x \in \Omega$;
- (h_7) there exist open sets ω and ω_1 , with $\bar{\omega} \subset \omega_1 \subseteq \Omega$, such that

$$\limsup_{s \rightarrow 0^+} \frac{\int_{\omega} F(x, s) dx}{s^2} = +\infty$$

and

$$\liminf_{s \rightarrow 0^+} \frac{\int_{\omega_1 \setminus \omega} F(x, s) dx}{s^2} > -\infty;$$

(h_{10}) there exist constants $q \in]1, \frac{N}{N-1}[$, $c_1 > 0$ and a function $c_2 \in L^{\frac{q}{q-1}}(\Omega)$ such that

$$|f(x, s)| \leq c_1 s^{q-1} + c_2(x)$$

for a.e. $x \in \Omega$ and every $s \in [0, +\infty[$;

(h_{11}) $\limsup_{s \rightarrow +\infty} \frac{F(x, s)}{s} \leq 0$ uniformly a.e. in Ω .

Then, for every $\lambda \in]0, +\infty[$, problem (1) has at least one positive solution.

Proof. Fix $\lambda > 0$. As we are assuming (h_5) , (h_{10}) and (h_{11}) , we can argue like in Step 1 and Step 2 of the proof of Theorem 3.2 to get a global minimizer u_λ of the action functional \mathcal{I}_λ , which is a solution of (1) satisfying $u_\lambda(x) \geq 0$ a.e. in Ω . To prove that u_λ is non-trivial, we exploit assumption (h_7) exactly as we did in Step 2 of the proof of Theorem 3.1, observing that (30) and (31) hold with G replaced by F . \square

In the next statement we just require F to be desultorily subquadratic at 0 and desultorily sublinear at $+\infty$.

Theorem 3.5. *Assume*

- (h_1) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{0,1}$ boundary $\partial\Omega$;
- (h_5) $f(x, 0) \geq 0$ for a.e. $x \in \Omega$;
- (h_7) there exist open sets ω and ω_1 , with $\bar{\omega} \subset \omega_1 \subseteq \Omega$, such that

$$\limsup_{s \rightarrow 0^+} \frac{\int_{\omega} F(x, s) dx}{s^2} = +\infty$$

and

$$\liminf_{s \rightarrow 0^+} \frac{\int_{\omega_1 \setminus \omega} F(x, s) dx}{s^2} > -\infty;$$

- (h_{13}) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the L^p -Carathéodory conditions for some $p > N$;
- (h_{14}) there exist a constant r and a continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x, s) \leq h(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \geq r$$

and

$$\liminf_{s \rightarrow +\infty} \frac{H(s)}{s} \leq 0,$$

where $H(s) = \int_0^s h(t) dt$.

Then, for every $\lambda \in]0, +\infty[$, problem (1) has at least one positive solution $u_\lambda \in BV(\Omega) \cap L^\infty(\Omega)$.

Proof. Fix $\lambda > 0$. As we are assuming (h_5) , (h_{13}) and (h_{14}) we can argue like in Step 1 of the proof of Theorem 3.3 to get a solution u_λ of (1), with $u_\lambda \in L^\infty(\Omega)$ and $u_\lambda(x) \geq 0$ a.e. in Ω . To prove that u_λ is non-trivial, we exploit assumption (h_7) exactly as we did in Step 2 of the proof of Theorem 3.1, observing that (30) and (31) hold with G replaced by F . \square

3.1.4. Potential quadratic at zero and linear at infinity

We discuss now the existence of positive solutions of the parameter independent problem

$$\begin{cases} -\operatorname{div}(\nabla u / \sqrt{1 + |\nabla u|^2}) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (48)$$

in the two limiting cases where the potential F may grow quadratically at 0 or linearly at $+\infty$. More precisely, we will replace the subquadraticity conditions at 0 with assumptions relating the behavior at 0 of $\frac{F(x,s)}{s^2}$ with the spectral constants λ_1 or λ_1^* , defined by (11) or (14), and the sublinearity condi-

tions at $+\infty$ with assumptions relating the behavior at $+\infty$ of $\frac{F(x,s)}{s}$ with the spectral constants μ_1 or μ_1^* , defined by (12) or (15).

Theorem 3.6. Assume

- (h₁) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{0,1}$ boundary $\partial\Omega$;
- (h₂) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions;
- (h₁₀) there exist constants $q \in]1, \frac{N}{N-1}[$, $c_1 > 0$ and a function $c_2 \in L^{\frac{q}{q-1}}(\Omega)$ such that

$$|f(x, s)| \leq c_1 s^{q-1} + c_2(x)$$

for a.e. $x \in \Omega$ and every $s \in [0, +\infty[$;

- (h₁₅) $\liminf_{s \rightarrow 0^+} \frac{2F(x,s)}{s^2} > \lambda_1$ uniformly a.e. in Ω , where λ_1 is defined by (11);

- (h₁₆) $\limsup_{s \rightarrow +\infty} \frac{F(x,s)}{s} < \mu_1$ uniformly a.e. in Ω , where μ_1 is defined by (12).

Then problem (48) has at least one positive solution.

Proof. We modify the function f , for a.e. $x \in \Omega$ and every $s < 0$, like in (35). Then, arguing as in Step 1 of the proof of Theorem 3.2 and using (h₁₆), we prove the existence of a solution u of (48), which is a global minimizer of the functional $\mathcal{I} = \mathcal{J} - \mathcal{F}$. To show that $u(x) \geq 0$ a.e. in Ω we proceed as in Step 2 of the proof of Theorem 3.2. Let us prove that u is non-trivial. By (h₁₅) there exists a constant $r > 0$ such that, for a.e. $x \in \Omega$ and every $s \in [0, r]$,

$$F(x, s) \geq \frac{\lambda_1}{2} s^2.$$

Denote by φ_1 the positive principal eigenfunction of $-\Delta$ in $H_0^1(\Omega)$ such that $\int_{\Omega} \varphi_1^2 dx = 1$. Since $\varphi_1 \in L^\infty(\Omega)$ (see [19, Theorem 8.15]), there exists $\varepsilon > 0$ such that $\varepsilon \|\varphi_1\|_{L^\infty(\Omega)} \leq r$. Then we have

$$\begin{aligned} \mathcal{I}(\varepsilon\varphi_1) &= \int_{\Omega} \sqrt{1 + \varepsilon^2 |\nabla \varphi_1|^2} dx - \int_{\Omega} F(x, \varepsilon\varphi_1) dx \\ &< \frac{1}{2} \varepsilon^2 \left(\int_{\Omega} |\nabla \varphi_1|^2 dx - \lambda_1 \int_{\Omega} \varphi_1^2 dx \right) + \text{meas}(\Omega) = \mathcal{I}(0). \end{aligned}$$

This implies that $\mathcal{I}(u) < \mathcal{I}(0)$ and hence $u \neq 0$. \square

Remark 3.3. It is clear from this proof that in Theorem 3.6, instead of (h₁₅), it is sufficient to assume

- (h₁₇) there exists a constant $r > 0$ such that, for a.e. $x \in \Omega$ and every $s \in [0, r]$,

$$\frac{2F(x, s)}{s^2} \geq \lambda_1,$$

where λ_1 is defined by (11).

The following result is a variant of Theorem 3.6 where only a desultory quadratic growth at 0 and a desultory linear growth at $+\infty$ are assumed on F .

Theorem 3.7. Assume

- (h_1) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{0,1}$ boundary $\partial\Omega$;
 (h_{13}) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the L^p -Carathéodory conditions for some $p > N$;
 (h_{18}) there exist a constant $r_0 > 0$ and a continuous function $k : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x, s) \geq k(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in [0, r_0],$$

$$\liminf_{s \rightarrow 0^+} \frac{K(s)}{s^2} \geq 0$$

and

$$\limsup_{s \rightarrow 0^+} \frac{2K(s)}{s^2} > \lambda_1^\sharp,$$

where $K(s) = \int_0^s k(t) dt$ and λ_1^\sharp is defined by (16);

- (h_{19}) there exist a constant $r_1 > 0$ and a continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x, s) \leq h(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \geq r_1$$

and

$$\liminf_{s \rightarrow +\infty} \frac{H(s)}{s} < \mu_1^*,$$

where $H(s) = \int_0^s h(t) dt$ and μ_1^* is defined by (15).

Then problem (48) has at least one positive solution.

Proof. Condition (h_{18}) implies that $f(x, 0) \geq 0$ for a.e. $x \in \Omega$. Then, by (h_{13}), we have that $\alpha = 0$ is a lower solution of (48). As (h_{19}) holds, we can apply the claim in the proof of Theorem 3.3, with $g = f$ and $\ell = h$, to get an upper solution $\beta \in C^2(\bar{\Omega})$ of (48) such that $\min_{\bar{\Omega}} \beta \geq r_0$. By Proposition 2.1 there exists a solution u of (48) such that $0 \leq u(x) \leq \beta(x)$ a.e. in Ω and

$$\mathcal{I}(u) = \min \{ \mathcal{I}(v) \mid v \in BV(\Omega), 0 \leq v(x) \leq \beta(x) \text{ a.e. in } \Omega \}.$$

We show that u is non-trivial by producing a function $w \in BV(\Omega)$ such that $0 \leq w(x) \leq \beta(x)$ a.e. in Ω and $\mathcal{I}(w) < \mathcal{I}(0)$. Let $R(\Omega) > 0$ be the largest $R > 0$ such that there is an open ball of radius R contained in Ω . Let B_2 be a ball of center x_0 and radius $R(\Omega)$, such that $B_2 \subset \Omega$, and let B_1 be the ball of center x_0 and radius $\frac{R(\Omega)}{2}$. Let $w \in H_0^1(\Omega)$ be the function defined by

$$w(x) = \max \left\{ 0, \min \left\{ 1, 2 \left(1 - \frac{\|x - x_0\|}{R(\Omega)} \right) \right\} \right\}.$$

By assumption (h_{18}) there exist $\varepsilon > 0$ and a sequence $(d_n)_n$ such that $\lim_{n \rightarrow +\infty} d_n = 0$, $d_n \leq r_0$, $K(d_n) > \frac{1}{2}(\lambda_1^\sharp + \varepsilon)d_n^2$ and $K(d_n w(x)) \geq -\frac{1}{4}\varepsilon \frac{\text{meas}(B_1)}{\text{meas}(B_2 \setminus B_1)} d_n^2$ for every n and a.e. $x \in \Omega$. Then we have

$$\mathcal{I}(d_n w) = \int_{\Omega} \sqrt{1 + d_n^2 |\nabla w|^2} dx - \int_{\Omega} F(x, d_n w) dx$$

$$\begin{aligned}
&\leq \frac{1}{2} d_n^2 \int_{B_2 \setminus B_1} |\nabla w|^2 dx + \text{meas}(\Omega) - \int_{B_1} K(d_n) dx - \int_{B_2 \setminus B_1} K(d_n w) dx \\
&\leq \frac{1}{2} d_n^2 \left(\int_{B_2 \setminus B_1} |\nabla w|^2 dx - \lambda_1^\# \text{meas}(B_1) - \frac{1}{2} \varepsilon \text{meas}(B_1) \right) + \text{meas}(\Omega) \\
&= \frac{1}{2} d_n^2 \text{meas}(B_1) \left(\left(\frac{2}{R(\Omega)} \right)^2 \frac{\text{meas}(B_2 \setminus B_1)}{\text{meas}(B_1)} - \lambda_1^\# - \frac{1}{2} \varepsilon \right) + \text{meas}(\Omega) \\
&= -\frac{1}{4} d_n^2 \varepsilon \text{meas}(B_1) + \text{meas}(\Omega) < \mathcal{I}(0).
\end{aligned}$$

This implies that $\mathcal{I}(u) < \mathcal{I}(0)$ and hence $u \neq 0$. \square

Remark 3.4. It is clear from the proof of Theorem 3.3 that in Theorem 3.7, instead of (h_{19}) , it is sufficient to assume

(h_{20}) there exist $R \geq \frac{1}{\mu_1^*}$ and $h : \mathbb{R} \rightarrow]0, +\infty[$ continuous such that

$$f(x, s) \leq h(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \left[R - \frac{1}{\mu_1^*}, R \right]$$

and

$$\frac{H(R) - H(s)}{R - s} \leq \mu_1^* \quad \text{for every } s \in \left[R - \frac{1}{\mu_1^*}, R \right],$$

where $H(s) = \int_0^s h(t) dt$ and μ_1^* is defined by (15).

3.1.5. Potential superquadratic at zero

The relevant assumption in this context is (h_{22}) . The following theorem is a slightly more general version of a result first obtained in [11]. Its proof is given in [37] by a hopefully more transparent argument than the original one.

Theorem 3.8. Assume

(h_{21}) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{1,1}$ boundary $\partial\Omega$;

(h_2) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions;

(h_{22}) there exist constants $r > 0$, $c > 1$ and $q > 2$, with $q < \frac{2N}{N-2}$ if $N \geq 3$, such that

$$s^{q-1} \leq f(x, s) \leq cs^{q-1}$$

for a.e. $x \in \Omega$ and every $s \in [0, r]$;

(h_{23}) there exist constants $r > 0$ and $\sigma \in]0, \frac{1}{2}[$ such that

$$F(x, s) \leq \sigma s f(x, s) \tag{49}$$

for a.e. $x \in \Omega$ and every $s \in [0, r]$;

(h_{24}) there exists a constant $r > 0$ such that

$$\frac{f(x, s)}{s} \leq \frac{f(x, t)}{t}$$

for a.e. $x \in \Omega$ and every $s, t \in]0, r]$, with $s < t$.

Then, for any given $p > N$, there exists $\lambda_* \in [0, +\infty[$ such that, for every $\lambda \in]\lambda_*, +\infty[$, problem (1) has at least one strong solution $u_\lambda \in W^{2,p}(\Omega)$, satisfying $u_\lambda(x) > 0$ for every $x \in \Omega$ and $\frac{\partial u_\lambda}{\partial \nu}(x) < 0$ for every $x \in \partial\Omega$, ν being the unit outer normal to Ω at $x \in \partial\Omega$,

$$\mathcal{I}_\lambda(u_\lambda) > \mathcal{I}_\lambda(0) = \text{meas}(\Omega) \quad \text{and} \quad \lim_{\lambda \rightarrow +\infty} \|u_\lambda\|_{W^{2,p}(\Omega)} = 0.$$

Proof. A detailed proof of this theorem can be found in [37]. \square

Remark 3.5. Assumption (h_{24}) implies $f(x, \cdot)$ increasing in $[0, r]$ for a.e. $x \in \Omega$; moreover it implies (49) with $\sigma = \frac{1}{2}$. Condition (h_{22}) implies $\frac{1}{\sigma} \leq q$.

3.1.6. Potential superlinear at infinity

The relevant assumption in this context is (h_{25}) . The proof of the following theorem makes use of a regularization procedure inspired from [43]. Related results can be found in [25] and [28].

Theorem 3.9. Assume

(h_1) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{0,1}$ boundary $\partial\Omega$;

(h_2) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions;

(h_5) $f(x, 0) \geq 0$ for a.e. $x \in \Omega$;

(h_{10}) there exist constants $q \in]1, \frac{N}{N-1}[$, $c_1 > 0$ and a function $c_2 \in L^{\frac{q}{q-1}}(\Omega)$ such that

$$|f(x, s)| \leq c_1 s^{q-1} + c_2(x)$$

for a.e. $x \in \Omega$ and every $s \in [0, +\infty[$;

(h_{25}) there exist a constant $p \in]1, \frac{N}{N-1}[$ and a function $a_\infty \in L^\infty(\Omega)$, with $a_\infty(x) \geq 0$ a.e. in Ω and $a_\infty(x) > 0$ in a set of positive measure, such that

$$\liminf_{s \rightarrow +\infty} \frac{F(x, s)}{s^p} \geq a_\infty(x)$$

uniformly a.e. in Ω , i.e. for every $k > 0$ there exists $s_k > 0$ such that $F(x, s) \geq (a_\infty(x) - k)s^p$ for a.e. $x \in \Omega$ and every $s \geq s_k$;

(h_{26}) there exists a constant $\vartheta \in]0, 1[$ such that

$$\limsup_{s \rightarrow +\infty} \left(\frac{F(x, s)}{s} - \vartheta f(x, s) \right) \leq 0$$

uniformly a.e. in Ω , i.e. for every $k > 0$ there is $s_k > 0$ such that $F(x, s) - \vartheta f(x, s)s \leq ks$ for a.e. $x \in \Omega$ and every $s \geq s_k$.

Then there exist $\lambda^* \in]0, +\infty[$ and $\eta > 0$ such that, for every $\lambda \in]0, \lambda^*[$, problem (1) has at least one positive solution u_λ , satisfying

$$\mathcal{I}_\lambda(u_\lambda) \geq \mathcal{I}_\lambda(0) = \text{meas}(\Omega) \quad \text{or} \quad \|u_\lambda\|_{L^q(\Omega)} \geq \eta \lambda^{-\frac{1}{q}},$$

where $q \in]1, \frac{N}{N-1}[$ is defined in (h_{10}) .

Proof. We modify f by setting $f(x, s) = f(x, 0)$ for a.e. $x \in \Omega$ and all $s < 0$. Hence assumptions (h_{10}) and (h_{26}) imply that there exist constants $q \in]1, \frac{N}{N-1}[$, $c_1 > 0$ and a function $c_2 \in L^{\frac{q}{q-1}}(\Omega)$ such that, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$,

$$|f(x, s)| \leq c_1 |s|^{q-1} + c_2(x) \quad (50)$$

and that there exists a constant $\vartheta \in]0, 1[$ such that

$$\limsup_{|s| \rightarrow +\infty} \left(\frac{F(x, s)}{|s|} - \vartheta f(x, s) \right) \leq 0 \quad (51)$$

uniformly a.e. in Ω .

Step 1. The elliptic regularization scheme. For each $\varepsilon > 0$ let us consider the regularized problem

$$\begin{cases} -\varepsilon \operatorname{div}(|\nabla u|^{r-2} \nabla u) - \operatorname{div}(\nabla u / \sqrt{1 + |\nabla u|^2}) = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (52)$$

where $r \in]1, \min\{p, \frac{1}{\vartheta}\}[$ is a fixed constant, $p \in]1, \frac{N}{N-1}[$ being defined in (h_{25}) and $\vartheta \in]0, 1[$ in (h_{26}) . Note that $p \leq q$, where q is defined in (h_{10}) . By a solution of (52) we mean a function $u \in W_0^{1,r}(\Omega)$ such that

$$\varepsilon \int_{\Omega} |\nabla u|^{r-2} \nabla u \nabla v \, dx + \int_{\Omega} \frac{\nabla u \nabla v}{\sqrt{1 + |\nabla u|^2}} \, dx = \lambda \int_{\Omega} f(x, u) v \, dx \quad (53)$$

for every $v \in W_0^{1,r}(\Omega)$. Let us define the functionals $\mathcal{J}_{\varepsilon} : W_0^{1,r}(\Omega) \rightarrow \mathbb{R}$ by

$$\mathcal{J}_{\varepsilon}(u) = \frac{\varepsilon}{r} \int_{\Omega} |\nabla u|^r \, dx + \int_{\Omega} \sqrt{1 + |\nabla u|^2} \, dx,$$

and $\mathcal{I}_{\lambda,\varepsilon} : W_0^{1,r}(\Omega) \rightarrow \mathbb{R}$ by

$$\mathcal{I}_{\lambda,\varepsilon}(u) = \mathcal{J}_{\varepsilon}(u) - \lambda \mathcal{F}(u).$$

The functionals $\mathcal{J}_{\varepsilon}$ and $\mathcal{I}_{\lambda,\varepsilon}$ are of class C^1 . Let $u \in W_0^{1,r}(\Omega)$ be a solution of (52). Since $\mathcal{J}_{\varepsilon}$ is convex we have

$$\mathcal{J}_{\varepsilon}(v) \geq \mathcal{J}_{\varepsilon}(u) + \mathcal{J}'_{\varepsilon}(u)(v - u)$$

for every $v \in W_0^{1,r}(\Omega)$. Testing against $v - u$ in (53), we get

$$\mathcal{J}'_{\varepsilon}(u)(v - u) = \lambda \int_{\Omega} f(x, u)(v - u) \, dx$$

and hence

$$\mathcal{J}_{\varepsilon}(v) \geq \mathcal{J}_{\varepsilon}(u) + \lambda \int_{\Omega} f(x, u)(v - u) \, dx. \quad (54)$$

Step 2. Mountain pass geometry for small $\lambda > 0$. Define

$$S = \{u \in W_0^{1,r}(\Omega) \mid \|\nabla u\|_{L^1(\Omega)} = 1\}.$$

Claim. There exist constants $\lambda_0 > 0$ and $c_0 > \text{meas}(\Omega)$ such that, for any $\lambda \in]0, \lambda_0]$, any $\varepsilon > 0$ and any $u \in S$,

$$\mathcal{I}_{\lambda,\varepsilon}(u) \geq c_0 > \mathcal{I}_{\lambda,\varepsilon}(0) = \text{meas}(\Omega). \quad (55)$$

Moreover, for each $\lambda > 0$ and $w \in W_0^{1,r}(\Omega)$, with $w(x) > 0$ a.e. in Ω , there exists $t = t_{\lambda,w} > 0$ such that, for any $\varepsilon \in]0, 1]$,

$$\|t \nabla w\|_{L^1(\Omega)} > 1 \quad \text{and} \quad \mathcal{I}_{\lambda,\varepsilon}(tw) < \mathcal{I}_{\lambda,\varepsilon}(0) = \text{meas}(\Omega). \quad (56)$$

Condition (50) implies that

$$|F(x, s)| \leq c_1 |s|^q + c_2(x) |s| \quad (57)$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. Using Jensen, Hölder and Poincaré inequalities, we see that there exists $c_0 > \text{meas}(\Omega)$ such that, for every $u \in S$ and any $\varepsilon > 0$,

$$\begin{aligned} \mathcal{I}_{\lambda,\varepsilon}(u) &\geq \int_{\Omega} \sqrt{1 + |\nabla u|^2} dx - \lambda \int_{\Omega} c_2 |u| dx - \lambda c_1 \int_{\Omega} |u|^q dx \\ &\geq \text{meas}(\Omega) \sqrt{1 + \left(\frac{\int_{\Omega} |\nabla u| dx}{\text{meas}(\Omega)} \right)^2} - \lambda \|c_2\|_{L^{\frac{q}{q-1}}(\Omega)} \mu_q^{-1} \|\nabla u\|_{L^1(\Omega)} - \lambda c_1 \mu_q^{-q} \|\nabla u\|_{L^1(\Omega)}^q \\ &= \sqrt{\text{meas}(\Omega)^2 + 1} - \lambda \|c_2\|_{L^{\frac{q}{q-1}}(\Omega)} \mu_q^{-1} - \lambda c_1 \mu_q^{-q} \geq c_0 > \mathcal{I}_{\lambda,\varepsilon}(0) = \text{meas}(\Omega), \end{aligned}$$

for each $\lambda \in]0, \lambda_0]$, with $\lambda_0 > 0$ such that

$$\lambda_0 < \left(\sqrt{\text{meas}(\Omega)^2 + 1} - \text{meas}(\Omega) \right) \left(\|c_2\|_{L^{\frac{q}{q-1}}(\Omega)} \mu_q^{-1} + c_1 \mu_q^{-q} \right)^{-1}.$$

This yields the first conclusion of the claim. Next we note that (h_{25}) and (57) imply that, for every $k > 0$, there exists $\ell \in L^{\frac{q}{q-1}}(\Omega)$ such that

$$F(x, s) \geq (a_{\infty}(x) - k) s^p - \ell(x) \quad (58)$$

for a.e. $x \in \Omega$ and every $s \geq 0$. Fix $\lambda > 0$ and $w \in W_0^{1,r}(\Omega)$, with $w(x) > 0$ a.e. in Ω , and choose $k > 0$ such that $\int_{\Omega} (a_{\infty} - k) w^p dx > 0$. By (58) we get, for every $t \geq 1$ and every $\varepsilon \in]0, 1]$,

$$\mathcal{I}_{\lambda,\varepsilon}(tw) \leq \frac{t^r}{r} \int_{\Omega} |\nabla w|^r dx + t \int_{\Omega} \sqrt{1 + |\nabla w|^2} dx - t^p \lambda \int_{\Omega} (a_{\infty} - k) w^p dx + \lambda \int_{\Omega} \ell dx.$$

Since $p > r$ we derive

$$\lim_{t \rightarrow +\infty} \mathcal{I}_{\lambda,\varepsilon}(tw) = -\infty$$

uniformly with respect to $\varepsilon \in]0, 1]$. Accordingly, the last conclusion of the claim is achieved too.

Step 3. Mountain pass levels. We define, for each $\lambda \in]0, \lambda_0]$ and $\varepsilon \in]0, 1]$, the mountain pass level

$$c_{\lambda, \varepsilon} = \inf_{\gamma \in \Gamma_\lambda} \max_{\tau \in [0, 1]} \mathcal{I}_{\lambda, \varepsilon}(\gamma(\tau)) \geq c_0,$$

where

$$\Gamma_\lambda = \{\gamma \in C^0([0, 1], W_0^{1,r}(\Omega)) \mid \gamma(0) = 0, \gamma(1) = tw\}, \quad (59)$$

with t, w satisfying (56).

Step 4. Palais–Smale condition. Let $\lambda > 0$ and $\varepsilon > 0$ be fixed. Assume $(u_n)_n \subset W_0^{1,r}(\Omega)$ is a (PS) sequence, i.e.

$$\sup_n |\mathcal{I}_{\lambda, \varepsilon}(u_n)| < +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} \mathcal{I}'_{\lambda, \varepsilon}(u_n) = 0 \quad \text{in } (W_0^{1,r}(\Omega))^*.$$

Then there exist a subsequence of $(u_n)_n$, which we still denote by $(u_n)_n$, and $u \in W_0^{1,r}(\Omega)$ such that $\lim_{n \rightarrow +\infty} u_n = u$ in $W_0^{1,r}(\Omega)$. We first prove that $(u_n)_n$ is bounded in $W_0^{1,r}(\Omega)$. Since $(u_n)_n$ is a (PS) sequence we have that, for some $c > 0$ and any n large enough,

$$\mathcal{I}_{\lambda, \varepsilon}(u_n) = \frac{\varepsilon}{r} \int_{\Omega} |\nabla u_n|^r dx + \int_{\Omega} \sqrt{1 + |\nabla u_n|^2} dx - \lambda \int_{\Omega} F(x, u_n) dx \leq c$$

and

$$\begin{aligned} |\vartheta \mathcal{I}'_{\lambda, \varepsilon}(u_n)(u_n)| &= \left| \varepsilon \vartheta \int_{\Omega} |\nabla u_n|^r dx + \vartheta \int_{\Omega} \frac{|\nabla u_n|^2}{\sqrt{1 + |\nabla u_n|^2}} dx - \lambda \vartheta \int_{\Omega} f(x, u_n) u_n dx \right| \\ &\leq \|u_n\|_{W_0^{1,r}(\Omega)}, \end{aligned}$$

where ϑ comes from (51). Hence we get

$$\begin{aligned} &\varepsilon \left(\frac{1}{r} - \vartheta \right) \int_{\Omega} |\nabla u_n|^r dx + \int_{\Omega} \left(\sqrt{1 + |\nabla u_n|^2} - \vartheta \frac{|\nabla u_n|^2}{\sqrt{1 + |\nabla u_n|^2}} \right) dx \\ &\quad - \lambda \int_{\Omega} (F(x, u_n) - \vartheta f(x, u_n) u_n) dx \leq c + \|u_n\|_{W_0^{1,r}(\Omega)}, \end{aligned}$$

for all large n . By (51) and (50), for every $k > 0$ there exists $c_k \in L^{\frac{q}{q-1}}(\Omega)$ such that

$$F(x, s) - \vartheta f(x, s)s \leq k|s| + c_k(x) \quad (60)$$

for a.e. $x \in \Omega$ and every s . Using (19) with $\eta = \vartheta$, (60) and Poincaré inequality, we have for all large n

$$\begin{aligned} c + \|u_n\|_{W_0^{1,r}(\Omega)} &\geq \varepsilon \left(\frac{1}{r} - \vartheta \right) \|u_n\|_{W_0^{1,r}(\Omega)}^r + \frac{1}{2} (1 - \vartheta) \int_{\Omega} |\nabla u_n|^2 dx \\ &\quad + (1 - d) \text{meas}(\Omega) - \lambda k \int_{\Omega} |u_n| dx - \lambda \int_{\Omega} c_k dx \end{aligned}$$

$$\begin{aligned} &\geq \varepsilon \left(\frac{1}{r} - \vartheta \right) \|u_n\|_{W_0^{1,r}(\Omega)}^r + \left(\frac{1}{2}(1 - \vartheta) - \lambda k \mu_1^{-1} \right) \int_{\Omega} |\nabla u_n| dx \\ &\quad + (1 - d) \text{meas}(\Omega) - \lambda \int_{\Omega} c_k dx. \end{aligned}$$

Hence, taking $k > 0$ small enough, we can find $K > 0$ such that

$$\varepsilon \left(\frac{1}{r} - \vartheta \right) \|u_n\|_{W_0^{1,r}(\Omega)}^r \leq K + \|u_n\|_{W_0^{1,r}(\Omega)}.$$

As $1 < r < \frac{1}{\vartheta}$, we conclude that $(u_n)_n$ is bounded in $W_0^{1,r}(\Omega)$.

Passing to a subsequence if necessary, we may assume that $(u_n)_n$ converges weakly in $W_0^{1,r}(\Omega)$ to some function $u \in W_0^{1,r}(\Omega)$. As $q < \frac{N}{N-1} < \frac{rN}{N-r}$ and hence $W_0^{1,r}(\Omega)$ is compactly embedded into $L^q(\Omega)$, we may further assume that $(u_n)_n$ converges to u in $L^q(\Omega)$. The strong convergence in $W_0^{1,r}(\Omega)$ of $(u_n)_n$ to u will follow from [7, Lemma 3]. To this end we define the generalized Dirichlet form

$$a_\varepsilon(u, v) = \varepsilon \int_{\Omega} |\nabla u|^{r-2} \nabla u \nabla v dx + \int_{\Omega} \frac{\nabla u \nabla v}{\sqrt{1 + |\nabla u|^2}} dx,$$

for $u, v \in W_0^{1,r}(\Omega)$, and we observe that all hypotheses of [7, Lemma 3] are satisfied. Hence condition (S) therein will guarantee that $(u_n)_n$ converges to u strongly in $W_0^{1,r}(\Omega)$, if we show that

$$\lim_{n \rightarrow +\infty} (a_\varepsilon(u_n, u_n - u) - a_\varepsilon(u, u_n - u)) = 0.$$

We have

$$\begin{aligned} \lim_{n \rightarrow +\infty} a_\varepsilon(u_n, u_n - u) &= \lim_{n \rightarrow +\infty} \left(\varepsilon \int_{\Omega} |\nabla u_n|^{r-2} \nabla u_n (\nabla u_n - \nabla u) dx + \int_{\Omega} \frac{\nabla u_n (\nabla u_n - \nabla u)}{\sqrt{1 + |\nabla u_n|^2}} dx \right) \\ &= \lim_{n \rightarrow +\infty} (\mathcal{I}'_{\lambda, \varepsilon}(u_n)(u_n - u) + \lambda \mathcal{F}'(u_n)(u_n - u)) = 0. \end{aligned}$$

Indeed, as $\lim_{n \rightarrow +\infty} \mathcal{I}'_{\lambda, \varepsilon}(u_n) = 0$ in $(W_0^{1,r}(\Omega))^*$ and $(u_n)_n$ is bounded in $W_0^{1,r}(\Omega)$, we see that

$$\lim_{n \rightarrow +\infty} \mathcal{I}'_{\lambda, \varepsilon}(u_n)(u_n - u) = 0.$$

Further, as $\mathcal{F} : L^q(\Omega) \rightarrow \mathbb{R}$ is of class C^1 and $\lim_{n \rightarrow +\infty} u_n = u$ in $L^q(\Omega)$, we easily get

$$\lim_{n \rightarrow +\infty} \mathcal{F}'(u_n)(u_n - u) = 0.$$

We also have

$$\lim_{n \rightarrow +\infty} a_\varepsilon(u, u_n - u) = \lim_{n \rightarrow +\infty} (\mathcal{I}'_{\lambda, \varepsilon}(u)(u_n - u) + \lambda \mathcal{F}'(u)(u_n - u)) = 0.$$

Indeed, as $\mathcal{I}'_{\lambda,\varepsilon}(u) \in (W_0^{1,r}(\Omega))^*$ and $\lim_{n \rightarrow +\infty} u_n = u$ weakly in $W_0^{1,r}(\Omega)$, we see that

$$\lim_{n \rightarrow +\infty} \mathcal{I}'_{\lambda,\varepsilon}(u)(u_n - u) = 0.$$

Finally, as $\mathcal{F}'(u) : L^q(\Omega) \rightarrow \mathbb{R}$ is continuous and $\lim_{n \rightarrow +\infty} u_n = u$ in $L^q(\Omega)$, it follows that

$$\lim_{n \rightarrow +\infty} \mathcal{F}'(u)(u_n - u) = 0.$$

Step 5. Existence of solutions of the regularized problem. We are now in a position of proving the existence of solutions of (52), which are obtained as critical points of mountain pass type of the functional $\mathcal{I}_{\lambda,\varepsilon}$.

Claim. *There exist constants $\lambda_0 > 0$ and $c_0 > \text{meas}(\Omega)$ such that, for each $\lambda \in]0, \lambda_0]$ and each $\varepsilon \in]0, 1]$, the functional $\mathcal{I}_{\lambda,\varepsilon}$ has a critical point $u_{\lambda,\varepsilon}$, which is a non-trivial non-negative solution of (52), satisfying*

$$\mathcal{I}_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}) \geq c_0. \quad (61)$$

Further, for each $\lambda \in]0, \lambda_0]$, there is a constant $k_1 > 0$ such that, for each $\varepsilon \in]0, 1]$,

$$\mathcal{I}_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}) \leq k_1. \quad (62)$$

Fix $\lambda \in]0, \lambda_0]$, where λ_0 has been obtained in Step 2, and $\varepsilon \in]0, 1]$. The existence of a non-trivial critical point $u_{\lambda,\varepsilon}$ of $\mathcal{I}_{\lambda,\varepsilon}$, with

$$\mathcal{I}_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}) = c_{\lambda,\varepsilon} \geq c_0 > \mathcal{I}_{\lambda,\varepsilon}(0),$$

follows from Steps 2, 3, 4 and the mountain pass theorem (see e.g. [14, Theorem 5.7]). Testing (53) against $-u_{\lambda,\varepsilon}^- \in W_0^{1,r}(\Omega)$, we get

$$\varepsilon \int_{\Omega} |\nabla u_{\lambda,\varepsilon}^-|^r dx + \int_{\Omega} \frac{|\nabla u_{\lambda,\varepsilon}^-|^2}{\sqrt{1 + |\nabla u_{\lambda,\varepsilon}^-|^2}} dx = -\lambda \int_{\Omega} f(x, u_{\lambda,\varepsilon}^-) u_{\lambda,\varepsilon}^- dx.$$

As $f(x, s) \geq 0$ for a.e. $x \in \Omega$ and every $s \leq 0$, we conclude that $u_{\lambda,\varepsilon}^- = 0$, that is $u_{\lambda,\varepsilon}(x) \geq 0$ a.e. in Ω .

Finally estimate (62) follows from the observation that

$$\mathcal{I}_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}) = c_{\lambda,\varepsilon} = \inf_{\gamma \in \Gamma_{\lambda}} \max_{\tau \in [0,1]} \mathcal{I}_{\lambda,\varepsilon}(\gamma(\tau)) \leq \inf_{\gamma \in \Gamma_{\lambda}} \max_{\tau \in [0,1]} \mathcal{I}_{\lambda,1}(\gamma(\tau)) = c_{\lambda,1},$$

where Γ_{λ} has been defined in (59), by setting $k_1 = c_{\lambda,1}$.

Step 6. Norm estimates on the solutions of the regularized problem. We want to prove that, for each $\lambda \in]0, \lambda_0]$, there is a constant $k_2 > 0$ such that, for each $\varepsilon \in]0, 1]$ and any solution $u_{\lambda,\varepsilon}$ of (52) satisfying (62), we have

$$\|u_{\lambda,\varepsilon}\|_{W_0^{1,1}(\Omega)} \leq k_2. \quad (63)$$

Fix $\lambda \in]0, \lambda_0]$ and $\varepsilon \in]0, 1]$. Let $u_{\lambda,\varepsilon}$ be a solution of (52) satisfying (62). We have

$$\varepsilon \int_{\Omega} |\nabla u_{\lambda,\varepsilon}|^r dx + \int_{\Omega} \frac{|\nabla u_{\lambda,\varepsilon}|^2}{\sqrt{1 + |\nabla u_{\lambda,\varepsilon}|^2}} dx - \lambda \int_{\Omega} f(x, u_{\lambda,\varepsilon}) u_{\lambda,\varepsilon} dx = 0$$

and

$$\frac{\varepsilon}{r} \int_{\Omega} |\nabla u_{\lambda,\varepsilon}|^r dx + \int_{\Omega} \sqrt{1 + |\nabla u_{\lambda,\varepsilon}|^2} dx - \lambda \int_{\Omega} F(x, u_{\lambda,\varepsilon}) \leq k_1.$$

We know that for every $k > 0$ there exists $c_k \in L^{\frac{q}{q-1}}(\Omega)$ such that (60) holds. Using (19) with $\eta = \vartheta$, (60) and the Poincaré inequality, we obtain

$$\begin{aligned} k_1 &\geq \varepsilon \left(\frac{1}{r} - \vartheta \right) \int_{\Omega} |\nabla u_{\lambda,\varepsilon}|^r dx + \int_{\Omega} \left(\sqrt{1 + |\nabla u_{\lambda,\varepsilon}|^2} - \vartheta \frac{|\nabla u_{\lambda,\varepsilon}|^2}{\sqrt{1 + |\nabla u_{\lambda,\varepsilon}|^2}} \right) dx \\ &\quad - \lambda \int_{\Omega} (F(x, u_{\lambda,\varepsilon}) - \vartheta f(x, u_{\lambda,\varepsilon}) u_{\lambda,\varepsilon}) dx \\ &\geq \frac{1}{2} (1 - \vartheta) \int_{\Omega} |\nabla u_{\lambda,\varepsilon}| dx + (1 - d) \text{meas}(\Omega) - \lambda k \int_{\Omega} |u_{\lambda,\varepsilon}| dx - \lambda \int_{\Omega} c_k dx \\ &\geq \left(\frac{1}{2} (1 - \vartheta) - \lambda k \mu^{-1} \right) \int_{\Omega} |\nabla u_{\lambda,\varepsilon}| dx + (1 - d) \text{meas}(\Omega) - \lambda \int_{\Omega} c_k dx. \end{aligned}$$

This yields the existence of a constant $k_2 > 0$ such that (63) holds, for any $\varepsilon \in]0, 1]$.

Step 7. Convergence of the regularization scheme. Let $(\varepsilon_n)_n \subset]0, 1]$ be such that $\lim_{n \rightarrow +\infty} \varepsilon_n = 0$ and, for any fixed $\lambda \in]0, \lambda_0]$, let $u_n = u_{\lambda,\varepsilon_n}$ be a solution of (52) such that (61) and (63) hold. We know that u_n satisfies (54), that is

$$\begin{aligned} &\frac{\varepsilon_n}{r} \int_{\Omega} |\nabla w|^r dx + \int_{\Omega} \sqrt{1 + |\nabla w|^2} dx - \lambda \int_{\Omega} f(x, u_n) w dx \\ &\geq \frac{\varepsilon_n}{r} \int_{\Omega} |\nabla u_n|^r dx + \int_{\Omega} \sqrt{1 + |\nabla u_n|^2} dx - \lambda \int_{\Omega} f(x, u_n) u_n dx \end{aligned} \quad (64)$$

for every $w \in W_0^{1,r}(\Omega)$. Since

$$\|u_n\|_{BV(\Omega)} = \|u_n\|_{W_0^{1,1}(\Omega_0)} \leq k_2$$

for every n , by [20, Theorem 1.19, Theorem 14.2] there exists a subsequence of $(u_n)_n$, which we still denote by $(u_n)_n$, and a function $u \in BV(\Omega)$ such that $\lim_{n \rightarrow +\infty} u_n = u$ in $L^q(\Omega)$ and

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} \sqrt{1 + |\nabla u_n|^2} dx \geq \int_{\Omega} \sqrt{1 + |Du|^2} + \int_{\partial\Omega} |u|_{\partial\Omega} d\mathcal{H}_{N-1}.$$

As $\mathcal{F} : L^q(\Omega) \rightarrow \mathbb{R}$ is of class C^1 , we easily get

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f(x, u_n) u_n dx = \lim_{n \rightarrow +\infty} \mathcal{F}'(u_n)(u_n) = \mathcal{F}'(u)(u) = \int_{\Omega} f(x, u) u dx$$

and

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f(x, u_n) w \, dx = \lim_{n \rightarrow +\infty} \mathcal{F}'(u_n)(w) = \mathcal{F}'(u)(w) = \int_{\Omega} f(x, u) w \, dx$$

for every $w \in L^q(\Omega)$. Letting $n \rightarrow +\infty$ in (64), we obtain for every $w \in W_0^{1,r}(\Omega)$

$$\begin{aligned} & \mathcal{J}(w) - \lambda \int_{\Omega} f(x, u) w \, dx \\ &= \int_{\Omega} \sqrt{1 + |\nabla w|^2} \, dx - \lambda \lim_{n \rightarrow +\infty} \int_{\Omega} f(x, u_n) w \, dx \\ &\geq \liminf_{n \rightarrow +\infty} \frac{\varepsilon_n}{r} \int_{\Omega} |\nabla u_n|^r \, dx + \liminf_{n \rightarrow +\infty} \int_{\Omega} \sqrt{1 + |\nabla u_n|^2} \, dx - \lambda \lim_{n \rightarrow +\infty} \int_{\Omega} f(x, u_n) u_n \, dx \\ &\geq \liminf_{n \rightarrow +\infty} \frac{\varepsilon_n}{r} \int_{\Omega} |\nabla u_n|^r \, dx + \int_{\Omega} \sqrt{1 + |Du|^2} \, dx + \int_{\partial\Omega} |u|_{\partial\Omega} \, d\mathcal{H}_{N-1} - \lambda \int_{\Omega} f(x, u) u \, dx \\ &\geq \mathcal{J}(u) - \lambda \int_{\Omega} f(x, u) u \, dx. \end{aligned} \tag{65}$$

Since $W_0^{1,r}(\Omega)$ is dense in $W_0^{1,1}(\Omega)$, $\mathcal{J} : W_0^{1,1}(\Omega) \rightarrow \mathbb{R}$ is continuous and $\mathcal{F} : L^q(\Omega) \rightarrow \mathbb{R}$ is of class C^1 , we see that

$$\mathcal{J}(w) - \lambda \int_{\Omega} f(x, u) w \, dx \geq \mathcal{J}(u) - \lambda \int_{\Omega} f(x, u) u \, dx$$

for every $w \in W_0^{1,1}(\Omega)$. Fix $v \in BV(\Omega)$. By the approximation property in $BV(\Omega)$ (see Section 2) there exists a sequence $(w_n)_n \subset W_0^{1,1}(\Omega)$ such that $\lim_{n \rightarrow +\infty} w_n = v$ in $L^q(\Omega)$ and

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \sqrt{1 + |\nabla w_n|^2} \, dx = \int_{\Omega} \sqrt{1 + |Dv|^2} \, dx + \int_{\partial\Omega} |v|_{\partial\Omega} \, d\mathcal{H}_{N-1}.$$

Further we have

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f(x, u) w_n \, dx = \int_{\Omega} f(x, u) v \, dx.$$

This implies that

$$\mathcal{J}(v) - \lambda \int_{\Omega} f(x, u) v \, dx \geq \mathcal{J}(u) - \lambda \int_{\Omega} f(x, u) u \, dx.$$

Therefore (2) holds for every $v \in BV(\Omega)$, which means that u is a solution of (1).

Since, for each n , u_n is a non-negative solution of (52), we have $u(x) \geq 0$ in Ω . Let us prove that $u(x) > 0$ on a set of positive measure. Assume by contradiction that $u(x) = 0$ a.e. in Ω . As $\lim_{n \rightarrow +\infty} u_n = u = 0$ in $L^q(\Omega)$ and $\mathcal{F} : L^q(\Omega) \rightarrow \mathbb{R}$ is of class C^1 , we have

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f(x, u_n) u_n dx = 0 = \lim_{n \rightarrow +\infty} \int_{\Omega} F(x, u_n) dx.$$

Taking $w = 0$ in (64), we get for each n

$$\text{meas}(\Omega) \leq \frac{\varepsilon_n}{r} \int_{\Omega} |\nabla u_n|^r dx + \int_{\Omega} \sqrt{1 + |\nabla u_n|^2} dx \leq \lambda \int_{\Omega} f(x, u_n) u_n dx + \text{meas}(\Omega)$$

and hence

$$\lim_{n \rightarrow +\infty} \left(\frac{\varepsilon_n}{r} \int_{\Omega} |\nabla u_n|^r dx + \int_{\Omega} \sqrt{1 + |\nabla u_n|^2} dx \right) = \text{meas}(\Omega).$$

This yields

$$\lim_{n \rightarrow +\infty} \mathcal{I}_{\lambda, \varepsilon_n}(u_n) = \lim_{n \rightarrow +\infty} \left(\frac{\varepsilon_n}{r} \int_{\Omega} |\nabla u_n|^r dx + \int_{\Omega} \sqrt{1 + |\nabla u_n|^2} dx - \lambda \int_{\Omega} F(x, u_n) dx \right) = \text{meas}(\Omega),$$

thus contradicting (61), as $c_0 > \text{meas}(\Omega)$.

Step 8. Behavior of the solutions as $\lambda \rightarrow 0^+$. We want to prove that there exists $\eta > 0$ such that, for each $\lambda \in]0, \lambda_0]$, a positive solution u_λ of (1) can be selected so that

$$\mathcal{I}_\lambda(u_\lambda) \geq \mathcal{I}_\lambda(0) = \text{meas}(\Omega) \quad \text{or} \quad \|u_\lambda\|_{L^q(\Omega)} \geq \eta \lambda^{-\frac{1}{q}}.$$

Fix $\lambda \in]0, \lambda_0]$ and let u_λ be a solution of (1) as obtained in the preceding steps. Suppose $\mathcal{I}_\lambda(u_\lambda) < \mathcal{I}_\lambda(0)$, then

$$\mathcal{F}(u_\lambda) > 0. \tag{66}$$

Let $(u_n)_n$ be a sequence of solutions of (52) such that $\lim_{n \rightarrow +\infty} u_n = u_\lambda$ in $L^q(\Omega)$. Taking $w = 0$ in (64), we get for every n

$$\int_{\Omega} \sqrt{1 + |\nabla u_n|^2} dx \leq \lambda \int_{\Omega} f(x, u_n) u_n dx + \text{meas}(\Omega). \tag{67}$$

Arguing as in Step 7 and possibly passing to a subsequence, we have from (65)

$$\mathcal{J}(w) - \lambda \int_{\Omega} f(x, u_\lambda) w dx \geq \lim_{n \rightarrow +\infty} \frac{\varepsilon_n}{r} \int_{\Omega} |\nabla u_n|^r dx + \mathcal{J}(u_\lambda) - \lambda \int_{\Omega} f(x, u_\lambda) u_\lambda dx$$

for every $w \in W_0^{1,r}(\Omega)$ and hence

$$\mathcal{J}(v) - \lambda \int_{\Omega} f(x, u_\lambda) v dx \geq \lim_{n \rightarrow +\infty} \frac{\varepsilon_n}{r} \int_{\Omega} |\nabla u_n|^r dx + \mathcal{J}(u_\lambda) - \lambda \int_{\Omega} f(x, u_\lambda) u_\lambda dx$$

for every $v \in BV(\Omega)$. Testing against u_λ , we conclude that

$$\lim_{n \rightarrow +\infty} \frac{\varepsilon_n}{r} \int_{\Omega} |\nabla u_n|^r dx = 0. \quad (68)$$

As $\mathcal{F} : L^q(\Omega) \rightarrow \mathbb{R}$ is continuous, by (66) we have $\mathcal{F}(u_n) > 0$ for all large n . Using (61) we also get for every n

$$\frac{\varepsilon_n}{r} \int_{\Omega} |\nabla u_n|^r dx + \int_{\Omega} \sqrt{1 + |\nabla u_n|^2} dx \geq c_0, \quad (69)$$

where $c_0 > \text{meas}(\Omega)$ is a constant independent of $\lambda \in]0, \lambda_0]$. From (68) and (69) we get

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} \sqrt{1 + |\nabla u_n|^2} dx \geq c_0. \quad (70)$$

Now, letting $n \rightarrow +\infty$ in (67) and using (70), we get, as $\mathcal{F} : L^q(\Omega) \rightarrow \mathbb{R}$ is of class C^1 ,

$$\begin{aligned} \lambda \int_{\Omega} f(x, u_\lambda) u_\lambda dx &= \lambda \lim_{n \rightarrow +\infty} \int_{\Omega} f(x, u_n) u_n dx \\ &\geq \liminf_{n \rightarrow +\infty} \int_{\Omega} \sqrt{1 + |\nabla u_n|^2} dx - \text{meas}(\Omega) \geq c_0 - \text{meas}(\Omega) \end{aligned}$$

and then

$$\int_{\Omega} f(x, u_\lambda) u_\lambda dx \geq \frac{1}{\lambda} (c_0 - \text{meas}(\Omega)).$$

Finally, using (50), we get

$$\begin{aligned} \frac{1}{\lambda} (c_0 - \text{meas}(\Omega)) &\leq \int_{\Omega} f(x, u_\lambda) u_\lambda dx \leq c_1 \int_{\Omega} |u_\lambda|^q dx + \int_{\Omega} c_2 u_\lambda dx \\ &\leq c_1 \|u_\lambda\|_{L^q(\Omega)}^q + c_3 \|u_\lambda\|_{L^q(\Omega)} \end{aligned}$$

for some constant $c_3 > 0$.

Hence we conclude that there exist $\lambda^* \in]0, +\infty]$ and $\eta > 0$ such that, for every $\lambda \in]0, \lambda^*]$, problem (1) has at least one positive solution u_λ , satisfying $\mathcal{I}_\lambda(u_\lambda) \geq \mathcal{I}_\lambda(0) = \text{meas}(\Omega)$ or $\|u_\lambda\|_{L^q(\Omega)} \geq \eta \lambda^{-\frac{1}{q}}$. \square

Remark 3.6. Note that if we assume, in addition to all hypotheses of Theorem 3.9, that

(h₂₇) $F(x, s) \geq 0$ for a.e. $x \in \Omega$ and every $s \geq 0$,

then we conclude that $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_{L^q(\Omega)} = +\infty$.

3.1.7. Potential superquadratic at zero and superlinear at infinity

Combining Theorem 3.8 and Theorem 3.9 yields the following result. Unlike the one-dimensional case we discussed in [5], we are not able here to prove the existence of a positive solution for each $\lambda > 0$. It remains therefore an open question for us to know whether the intervals $]0, \lambda^*[$ and $] \lambda_*, +\infty[$ defined in the statement below overlap.

Theorem 3.10. Assume

- (h₂₁) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{1,1}$ boundary $\partial\Omega$;
 (h₂) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions;
 (h₂₂) there exist constants $r > 0$, $c > 1$ and $\rho > 2$, with $\rho < \frac{2N}{N-2}$ if $N \geq 3$, such that

$$s^{\rho-1} \leq f(x, s) \leq cs^{\rho-1}$$

for a.e. $x \in \Omega$ and every $s \in [0, r]$;

- (h₂₃) there exist constants $r > 0$ and $\sigma \in]0, \frac{1}{2}[$ such that

$$F(x, s) \leq \sigma s f(x, s)$$

for a.e. $x \in \Omega$ and every $s \in [0, r]$;

- (h₂₄) there exists a constant $r > 0$ such that

$$\frac{f(x, s)}{s} \leq \frac{f(x, t)}{t}$$

for a.e. $x \in \Omega$ and every $s, t \in]0, r]$, with $s < t$;

- (h₁₀) there exist constants $q \in]1, \frac{N}{N-1}[$, $c_1 > 0$ and a function $c_2 \in L^{\frac{q}{q-1}}(\Omega)$ such that

$$|f(x, s)| \leq c_1 s^{q-1} + c_2(x)$$

for a.e. $x \in \Omega$ and every $s \in [0, +\infty[$;

- (h₂₅) there exist a constant $p \in]1, \frac{N}{N-1}[$ and a function $a_\infty \in L^\infty(\Omega)$, with $a_\infty(x) \geq 0$ a.e. in Ω and $a_\infty(x) > 0$ in a set of positive measure, such that

$$\liminf_{s \rightarrow +\infty} \frac{F(x, s)}{s^p} \geq a_\infty(x)$$

uniformly a.e. in Ω ;

- (h₂₆) there exists a constant $\vartheta \in]0, 1[$ such that

$$\limsup_{s \rightarrow +\infty} \left(\frac{F(x, s)}{s} - \vartheta f(x, s) \right) \leq 0$$

uniformly a.e. in Ω .

Then there exist $\lambda_* \in [0, +\infty[$ and $\lambda^* \in]0, +\infty]$, such that, for every $\lambda \in]0, \lambda^*[\cup]\lambda_*, +\infty[$, problem (1) has at least one positive solution u_λ .

Proof. We combine Theorem 3.8 and Theorem 3.9. \square

3.2. Existence of at least two positive solutions

We now combine the existence results proved in the preceding section to obtain multiple solutions, which are distinguished according to their behavior as $\lambda \rightarrow 0^+$ or $\lambda \rightarrow +\infty$.

3.2.1. Potential subquadratic at zero and superlinear at infinity

The relevant assumptions here are the desultory subquadraticity condition (h_7) and the superlinearity condition (h_{25}) .

Theorem 3.11. Assume

- (h_4) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{1,\sigma}$ boundary $\partial\Omega$ for some $\sigma \in]0, 1]$;
- (h_2) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions;
- (h_5) $f(x, 0) \geq 0$ for a.e. $x \in \Omega$;
- (h_6) there exist constants $r > 0$ and $c > 0$ such that $|f(x, s)| \leq c$ for a.e. $x \in \Omega$ and every $s \in [0, r]$;
- (h_7) there exist open sets ω and ω_1 , with $\bar{\omega} \subset \omega_1 \subseteq \Omega$, such that

$$\limsup_{s \rightarrow 0^+} \frac{\int_{\omega} F(x, s) dx}{s^2} = +\infty$$

and

$$\liminf_{s \rightarrow 0^+} \frac{\int_{\omega_1 \setminus \omega} F(x, s) dx}{s^2} > -\infty;$$

- (h_{10}) there exist constants $q \in]1, \frac{N}{N-1}[$, $c_1 > 0$ and a function $c_2 \in L^{\frac{q}{q-1}}(\Omega)$ such that

$$|f(x, s)| \leq c_1 s^{q-1} + c_2(x)$$

for a.e. $x \in \Omega$ and every $s \in [0, +\infty[$;

- (h_{25}) there exist a constant $p \in]1, \frac{N}{N-1}[$ and a function $a_{\infty} \in L^{\infty}(\Omega)$, with $a_{\infty}(x) \geq 0$ a.e. in Ω and $a_{\infty}(x) > 0$ in a set of positive measure, such that

$$\liminf_{s \rightarrow +\infty} \frac{F(x, s)}{s^p} \geq a_{\infty}(x)$$

uniformly a.e. in Ω ;

- (h_{26}) there exists a constant $\vartheta \in]0, 1[$ such that

$$\limsup_{s \rightarrow +\infty} \left(\frac{F(x, s)}{s} - \vartheta f(x, s) \right) \leq 0$$

uniformly a.e. in Ω .

Then there exists $\lambda^* \in]0, +\infty]$ such that, for every $\lambda \in]0, \lambda^*[$, problem (1) has at least two positive solutions, one of which is a weak solution.

Proof. Let $\eta > 0$ be the constant defined in Theorem 3.9. By Theorem 3.1 we know that, for all small $\lambda > 0$, a positive solution $u_{\lambda}^{(1)}$ of (1) exists such that $\|u_{\lambda}^{(1)}\|_{L^q(\Omega)} < \eta$ and $\mathcal{I}_{\lambda}(u_{\lambda}^{(1)}) < \mathcal{I}_{\lambda}(0)$. On the other hand, by Theorem 3.9 we know that, for all small $\lambda > 0$, a positive solution $u_{\lambda}^{(2)}$ of (1) exists such that either $\|u_{\lambda}^{(2)}\|_{L^q(\Omega)} > \eta$ or $\mathcal{I}_{\lambda}(u_{\lambda}^{(2)}) \geq \mathcal{I}_{\lambda}(0)$. In particular we conclude that $u_{\lambda}^{(1)} \neq u_{\lambda}^{(2)}$. \square

3.2.2. Potential superquadratic at zero and sublinear at infinity

The relevant assumptions here are the superquadraticity condition (h_{22}) and the sublinearity condition (h_{11}) , in Theorem 3.12, and the desultory sublinearity condition (h_{14}) , in Theorem 3.13.

Theorem 3.12. Assume

(h_{21}) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{1,1}$ boundary $\partial\Omega$;

(h_2) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions;

(h_{22}) there exist constants $r > 0$, $c > 1$ and $\rho > 2$, with $\rho < \frac{2N}{N-2}$ if $N \geq 3$, such that

$$s^{\rho-1} \leq f(x, s) \leq cs^{\rho-1}$$

for a.e. $x \in \Omega$ and every $s \in [0, r]$;

(h_{23}) there exist constants $r > 0$ and $\sigma \in]0, \frac{1}{2}[$ such that

$$F(x, s) \leq \sigma s f(x, s)$$

for a.e. $x \in \Omega$ and every $s \in [0, r]$;

(h_{24}) there exists a constant $r > 0$ such that

$$\frac{f(x, s)}{s} \leq \frac{f(x, t)}{t}$$

for a.e. $x \in \Omega$ and every $s, t \in]0, r]$, with $s < t$;

(h_{10}) there exist constants $q \in]1, \frac{N}{N-1}[$, $c_1 > 0$ and a function $c_2 \in L^{\frac{q}{q-1}}(\Omega)$ such that

$$|f(x, s)| \leq c_1 s^{q-1} + c_2(x)$$

for a.e. $x \in \Omega$ and every $s \in [0, +\infty[$;

(h_{11})
$$\limsup_{s \rightarrow +\infty} \frac{F(x, s)}{s} \leq 0$$

uniformly a.e. in Ω .

Then there exists $\lambda_* \in [0, +\infty[$ such that, for every $\lambda \in]\lambda_*, +\infty[$, problem (1) has at least two positive solutions, one of which is a weak solution.

Proof. By Theorem 3.2 we know that, for all large $\lambda > 0$, a positive solution $u_\lambda^{(1)}$ of (1) exists such that $\liminf_{\lambda \rightarrow +\infty} \|u_\lambda^{(1)}\|_{L^q(\Omega)} > 0$. On the other hand, by Theorem 3.8 we know that, for all large $\lambda > 0$, a positive solution $u_\lambda^{(2)}$ of (1) exists such that $\lim_{\lambda \rightarrow +\infty} \|u_\lambda^{(2)}\|_{L^q(\Omega)} = 0$. In particular we have that $u_\lambda^{(1)} \neq u_\lambda^{(2)}$. \square

Theorem 3.13. Assume

(h_{21}) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{1,1}$ boundary $\partial\Omega$;

(h_{13}) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the L^p -Carathéodory conditions for some $p > N$;

(h_{22}) there exist constants $r > 0$, $c > 1$ and $q > 2$, with $q < \frac{2N}{N-2}$ if $N \geq 3$, such that

$$s^{q-1} \leq f(x, s) \leq cs^{q-1}$$

for a.e. $x \in \Omega$ and every $s \in [0, r]$;

(h₂₃) there exist constants $r > 0$ and $\sigma \in]0, \frac{1}{2}[$ such that

$$F(x, s) \leq \sigma s f(x, s)$$

for a.e. $x \in \Omega$ and every $s \in [0, r]$;

(h₂₄) there exists a constant $r > 0$ such that

$$\frac{f(x, s)}{s} \leq \frac{f(x, t)}{t}$$

for a.e. $x \in \Omega$ and every $s, t \in]0, r]$, with $s < t$;

(h₁₄) there exist a constant r and a continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x, s) \leq h(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \geq r$$

and

$$\liminf_{s \rightarrow +\infty} \frac{H(s)}{s} \leq 0,$$

where $H(s) = \int_0^s h(t) dt$.

Then there exists $\lambda_* \in [0, +\infty[$ such that, for every $\lambda \in]\lambda_*, +\infty[$, problem (1) has at least two positive solutions, one of which is a weak solution.

Proof. By Theorem 3.3 we know that, for all large $\lambda > 0$, a positive solution $u_\lambda^{(1)}$ of (1) exists such that $\mathcal{I}_\lambda(u_\lambda^{(1)}) < \mathcal{I}_\lambda(0) = \text{meas}(\Omega)$. On the other hand, by Theorem 3.8 we know that, for all large $\lambda > 0$, a positive solution $u_\lambda^{(2)}$ of (1) exists such that $\mathcal{I}_\lambda(u_\lambda^{(2)}) > \mathcal{I}_\lambda(0) = \text{meas}(\Omega)$. In particular we have that $u_\lambda^{(1)} \neq u_\lambda^{(2)}$. \square

3.3. Existence of at least three positive solutions

3.3.1. Potential superquadratic at zero and superlinear at infinity depending on two parameters

In this section we reconsider the case of a potential F which is superquadratic at 0 and superlinear at $+\infty$. The introduction of a second parameter into the equation will allow us to prove the existence of more solutions. Namely, let us consider the model two-parameters problem

$$\begin{cases} -\operatorname{div}(\nabla u / \sqrt{1 + |\nabla u|^2}) = \min\{\lambda(u^+)^{p-1}, \mu(u^+)^{q-1}\} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (71)$$

By performing a sharp analysis of the geometric features of the action functional associated with this problem, we can prove the existence of three solutions under some specific configurations of the parameters.

Theorem 3.14. Assume that

(h₄) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{1,\sigma}$ boundary $\partial\Omega$ for some $\sigma \in]0, 1]$;

and

(h₂₈) $p \in]1, \frac{N}{N-1}[$ and $q > 2$, with $q < \frac{2N}{N-2}$ if $N \geq 3$.

Then there exist $\lambda^* \in]0, +\infty[$ and a function $\mu_* :]0, \lambda^*[\rightarrow]0, +\infty[$ such that, for every $\lambda \in]0, \lambda^*[$ and $\mu \in]\mu_*(\lambda), +\infty[$, problem (71) has at least three positive solutions $u_{\lambda,\mu}^{(1)}, u_{\lambda,\mu}^{(2)}, u_{\lambda,\mu}^{(3)}$ with, for each $i = 2, 3$, $u_{\lambda,\mu}^{(i)} \in C^{1,\tau}(\bar{\Omega})$ for some $\tau \in]0, 1]$, $u_{\lambda,\mu}^{(i)}(x) > 0$ for every $x \in \Omega$ and $\frac{\partial u_{\lambda,\mu}^{(i)}}{\partial \nu}(x) < 0$ for every $x \in \partial\Omega$, ν being the unit outer normal at $x \in \partial\Omega$.

Proof. For each $\lambda, \mu > 0$, let us set

$$g_{\lambda,\mu}(s) = \min \left\{ (s^+)^{p-1}, \frac{\mu}{\lambda} (s^+)^{q-1} \right\}$$

and

$$G_{\lambda,\mu}(s) = \int_0^s g_{\lambda,\mu}(t) dt$$

for every $s \in \mathbb{R}$. Note that

$$G_{\lambda,\mu}(s) = \begin{cases} \frac{\mu}{\lambda} \frac{1}{q} (s^+)^q & \text{if } s \leq (\frac{\lambda}{\mu})^{\frac{1}{q-p}}, \\ \frac{1}{p} s^p - (\frac{1}{p} - \frac{1}{q}) (\frac{\lambda}{\mu})^{\frac{p}{q-p}} & \text{if } s > (\frac{\lambda}{\mu})^{\frac{1}{q-p}}. \end{cases}$$

Further, for any fixed $s_0 > 0$, we have $g_{\lambda,\mu}(s) = \frac{\mu}{\lambda} (s^+)^{q-1}$ in $]-\infty, s_0]$, if $\lambda, \mu > 0$ satisfy $s_0 \leq (\frac{\lambda}{\mu})^{\frac{1}{q-p}}$, and $g_{\lambda,\mu}(s) = s^{p-1}$ in $[s_0, +\infty[$, if $\lambda, \mu > 0$ satisfy $(\frac{\lambda}{\mu})^{\frac{1}{q-p}} \leq s_0$.

Step 1. There exists $\lambda_0 \in]0, +\infty[$ such that, for every $\lambda \in]0, \lambda_0[$ and every $\mu \geq \lambda$, problem (71) has at least one solution $u_{\lambda,\mu}^{(1)}$ with $\text{ess sup}_{\Omega} u_{\lambda,\mu}^{(1)} > 1$. The functions $g_{\lambda,\mu}$ and $G_{\lambda,\mu}$ satisfy conditions (h_2) , (h_5) , (h_{10}) , (h_{26}) and (h_{27}) , uniformly with respect to $\lambda, \mu > 0$. Moreover, for any fixed $\kappa_0 > 0$, condition (h_{25}) is fulfilled uniformly with respect to $\lambda, \mu > 0$ with $\frac{\lambda}{\mu} \leq \kappa_0$. Take $\kappa_0 = 1$. Theorem 3.9 and Remark 3.6 then yield the existence of $\lambda_0 \in]0, +\infty[$ such that, for every $\lambda \in]0, \lambda_0[$ and every $\mu \geq \lambda$, problem (71) has at least one solution $u_{\lambda,\mu}$ satisfying

$$\lim_{\lambda \rightarrow 0^+} \|u_{\lambda,\mu}\|_{L^p(\Omega)} = +\infty$$

uniformly with respect to $\mu \geq \lambda$. Possibly reducing λ_0 , we can suppose that $\|u_{\lambda,\mu}\|_{L^p(\Omega)} > \text{meas}(\Omega)$. Hence the conclusion follows.

Let us set $\mu_0 = 2^{\frac{q-p}{p}} \lambda_0$. If we assume $\lambda \in]0, \lambda_0[$ and $\mu \in]\mu_0, +\infty[$, then we have in particular $\mu > \lambda$ and hence $(\frac{\lambda}{\mu})^{\frac{1}{q-p}} < 1$.

Step 2. A modified problem. Like in the proof of Theorem 3.1, let $a : [0, +\infty[\rightarrow [0, +\infty[$ be the $C^{1,1}$ non-increasing function defined by (20) and let $A : [0, +\infty[\rightarrow [0, +\infty[$ be the potential of a defined by (21). Recall that the functions a and A satisfy conditions (22) and (23), respectively. As we already noticed, the structure and the regularity conditions assumed in [18] hold. Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a continuous function such that

$$\chi(s) = \begin{cases} 1 & \text{if } s \leq 1, \\ 0 & \text{if } s \geq 2. \end{cases}$$

For each $\lambda, \mu > 0$, we define

$$h_{\lambda,\mu}(s) = \chi(s) g_{\lambda,\mu}(s) \tag{72}$$

and

$$H_{\lambda,\mu}(s) = \int_0^s h_{\lambda,\mu}(t) dt,$$

for every $s \in \mathbb{R}$. Note that, for each $\lambda \in]0, \lambda_0[$ and $\mu \in]\mu_0, +\infty[$, we have

$$0 \leq h_{\lambda,\mu}(s) \leq 2^{p-1} \quad (73)$$

and

$$0 \leq H_{\lambda,\mu}(s) \leq \frac{2^p}{p} \quad (74)$$

for every $s \in \mathbb{R}$. Let us consider the modified problem

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^2)\nabla u) = \lambda h_{\lambda,\mu}(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (75)$$

A solution of (75) is a function $u \in H_0^1(\Omega)$ satisfying

$$\int_{\Omega} a(|\nabla u|^2) \nabla u \nabla v \, dx = \lambda \int_{\Omega} h_{\lambda,\mu}(u) v \, dx \quad (76)$$

for every $v \in H_0^1(\Omega)$. We define the functional $\mathcal{K}_{\lambda,\mu} : H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$\mathcal{K}_{\lambda,\mu}(u) = \frac{1}{2} \int_{\Omega} A(|\nabla u|^2) \, dx - \lambda \int_{\Omega} H_{\lambda,\mu}(u) \, dx.$$

$\mathcal{K}_{\lambda,\mu}$ is of class C^1 and weakly lower semicontinuous. Moreover, $u \in H_0^1(\Omega)$ is a solution of (75) if and only if u is a critical point of $\mathcal{K}_{\lambda,\mu}$.

Step 3. There exists a function $\mu_* :]0, \lambda_0[\rightarrow]\mu_0, +\infty[$ such that, for any $\lambda \in]0, \lambda_0[$ and $\mu \in]\mu_*(\lambda), +\infty[$, the functional $\mathcal{K}_{\lambda,\mu}$ has a global minimizer $u_{\lambda,\mu}^{(2)} \in H_0^1(\Omega)$ with $\mathcal{K}_{\lambda,\mu}(u_{\lambda,\mu}^{(2)}) < 0$. Fix any $\lambda \in]0, \lambda_0[$ and $\mu \in]\mu_0, +\infty[$. By (23) and (74) the functional $\mathcal{K}_{\lambda,\mu}$ is coercive and bounded from below in $H_0^1(\Omega)$ and hence it has a global minimizer $u_{\lambda,\mu}^{(2)} \in H_0^1(\Omega)$. Set $d_{\mu} = 2^{\frac{1}{p}} \left(\frac{\lambda_0}{\mu}\right)^{\frac{1}{q-p}}$. We have $d_{\mu} \in](\frac{\lambda}{\mu})^{\frac{1}{q-p}}, 1[$ and

$$\begin{aligned} H_{\lambda,\mu}(d_{\mu}) &= \frac{1}{p} d_{\mu}^p - \left(\frac{1}{p} - \frac{1}{q}\right) \left(\frac{\lambda}{\mu}\right)^{\frac{p}{q-p}} \\ &\geq \frac{1}{p} d_{\mu}^p - \frac{1}{p} \left(\frac{\lambda}{\mu}\right)^{\frac{p}{q-p}} \geq \frac{1}{p} \left[d_{\mu}^p - \frac{1}{2} \left(2^{\frac{1}{p}} \left(\frac{\lambda_0}{\mu}\right)^{\frac{1}{q-p}} \right)^p \right] \\ &= \frac{1}{p} \left[d_{\mu}^p - \frac{1}{2} d_{\mu}^p \right] = \frac{1}{2} \frac{1}{p} d_{\mu}^p. \end{aligned}$$

Take $w \in H_0^1(\Omega)$ such that $w(x) \geq 0$ in Ω and $w(x) = 1$ in some open subset ω of Ω . We get

$$\begin{aligned}
\mathcal{K}_{\lambda,\mu}(d_\mu w) &= \frac{1}{2} \int_{\Omega} A(d_\mu^2 |\nabla w|^2) dx - \lambda \int_{\omega} H_{\lambda,\mu}(d_\mu w) dx - \lambda \int_{\Omega \setminus \omega} H_{\lambda,\mu}(d_\mu w) dx \\
&\leq d_\mu^2 \left(\frac{1}{2} \int_{\Omega} |\nabla w|^2 dx - \lambda d_\mu^{-2} \int_{\omega} H_{\lambda,\mu}(d_\mu) dx \right) \\
&\leq \frac{1}{2} d_\mu^2 \left(\int_{\Omega} |\nabla w|^2 dx - \frac{\lambda}{p} \text{meas}(\omega) d_\mu^{p-2} \right) < 0,
\end{aligned}$$

provided $\lambda d_\mu^{p-2} > \frac{p}{\text{meas}(\omega)} \int_{\Omega} |\nabla w|^2 dx$, or equivalently

$$\mu > \mu_0 \left(\frac{p}{\text{meas}(\omega)} \int_{\Omega} |\nabla w|^2 dx \right)^{\frac{q-p}{2-p}} \lambda^{-\frac{q-p}{2-p}}.$$

This implies that

$$\mathcal{K}_{\lambda,\mu}(u_{\lambda,\mu}^{(2)}) = \min_{u \in H_0^1(\Omega)} \mathcal{K}_{\lambda,\mu}(u) < 0 \quad (77)$$

and hence in particular $u_{\lambda,\mu}^{(2)} \neq 0$. We finally define, for every $\lambda \in]0, \lambda_0[$,

$$\mu_*(\lambda) = \max \left\{ \mu_0, \mu_0 \left(\frac{p}{\text{meas}(\omega)} \int_{\Omega} |\nabla w|^2 dx \right)^{\frac{q-p}{2-p}} \lambda^{-\frac{q-p}{2-p}} \right\}.$$

Step 4. For each $\lambda \in]0, \lambda_0[$ and $\mu \in]\mu_*(\lambda), +\infty[$ the functional $\mathcal{K}_{\lambda,\mu}$ has a critical point $u_{\lambda,\mu}^{(3)} \in H_0^1(\Omega)$ with $\mathcal{K}_{\lambda,\mu}(u_{\lambda,\mu}^{(3)}) > 0$. We first prove that $\mathcal{K}_{\lambda,\mu}$ has a mountain pass geometry around 0. Fix $\lambda \in]0, \lambda_0[$ and $\mu \in]\mu_0, +\infty[$. By (h₂₈), thanks also to (23) and the fact that $H_{\lambda,\mu}(s) \leq \frac{\mu}{\lambda} \frac{|s|^q}{q}$ for all $s \in \mathbb{R}$, the continuous embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ implies the existence of a constant $c > 0$ such that

$$\begin{aligned}
\mathcal{K}_{\lambda,\mu}(u) &= \frac{1}{2} \int_{\Omega} A(|\nabla u|^2) dx - \lambda \int_{\Omega} H_{\lambda,\mu}(u) dx \\
&\geq \frac{7\sqrt{2}}{32} \int_{\Omega} |\nabla u|^2 dx - \frac{\mu}{q} \int_{\Omega} |u|^q dx \\
&\geq \|\nabla u\|_{L^2(\Omega)}^2 \left(\frac{7\sqrt{2}}{32} - \frac{\mu}{q} c \|\nabla u\|_{L^2(\Omega)}^{q-2} \right)
\end{aligned}$$

for every $u \in H_0^1(\Omega)$. Taking $r \in]0, (\frac{q}{\mu c} \frac{7\sqrt{2}}{32})^{\frac{1}{q-2}}[$ we have

$$\inf_{\|u\|_{H_0^1(\Omega)}=r} \mathcal{K}_{\lambda,\mu}(u) > 0. \quad (78)$$

Therefore, for each $\lambda \in]0, \lambda_0[$ and $\mu \in]\mu^*(\lambda), +\infty[$, we can take $r > 0$ such that (78) and (77) hold, with $\|u_{\lambda,\mu}^{(2)}\|_{H_0^1(\Omega)} > r$, that is $\mathcal{K}_{\lambda,\mu}$ has a mountain pass geometry around 0.

Next we prove that $\mathcal{K}_{\lambda,\mu}$ satisfies the Palais–Smale condition. Assume $(u_n)_n$ is a (PS) sequence in $H_0^1(\Omega)$, i.e.

$$\sup_n |\mathcal{K}_{\lambda,\mu}(u_n)| < +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} \mathcal{K}'_{\lambda,\mu}(u_n) = 0 \quad \text{in } H^{-1}(\Omega).$$

We want to prove that there exist a subsequence of $(u_n)_n$, which we still denote by $(u_n)_n$, and $u \in H_0^1(\Omega)$ such that $\lim_{n \rightarrow +\infty} u_n = u$. We first notice that, as $\mathcal{K}_{\lambda,\mu}$ is coercive, the sequence $(u_n)_n$ is bounded in $H_0^1(\Omega)$. Passing to a subsequence if necessary, we may assume that $(u_n)_n$ converges weakly in $H_0^1(\Omega)$ to some function $u \in H_0^1(\Omega)$. The strong convergence of $(u_n)_n$ to u in $H_0^1(\Omega)$ will follow from [7, Lemma 3]. To this end we define the generalized Dirichlet form

$$a(u, v) = \int_{\Omega} a(|\nabla u|^2) \nabla u \nabla v \, dx,$$

for $u, v \in H_0^1(\Omega)$, and we observe that all hypotheses of [7, Lemma 3] are satisfied. Hence condition (S) therein will guarantee that $(u_n)_n$ converges strongly to u if we show that

$$\lim_{n \rightarrow +\infty} (a(u_n, u_n - u) - a(u, u_n - u)) = 0.$$

We have

$$\begin{aligned} \lim_{n \rightarrow +\infty} a(u_n, u_n - u) &= \lim_{n \rightarrow +\infty} \int_{\Omega} a(|\nabla u_n|^2) \nabla u_n (\nabla u_n - \nabla u) \, dx \\ &= \lim_{n \rightarrow +\infty} \left(\mathcal{K}'_{\lambda,\mu}(u_n)(u_n - u) + \lambda \int_{\Omega} h_{\lambda,\mu}(u_n)(u_n - u) \, dx \right) = 0. \end{aligned}$$

Indeed, as $\lim_{n \rightarrow +\infty} \mathcal{K}'_{\lambda,\mu}(u_n) = 0$ in $H^{-1}(\Omega)$ and $(u_n)_n$ is bounded in $H_0^1(\Omega)$, we see that

$$\lim_{n \rightarrow +\infty} \mathcal{K}'_{\lambda,\mu}(u_n)(u_n - u) = 0.$$

Further, the compact embedding of $H_0^1(\Omega)$ into $L^1(\Omega)$ implies that $\lim_{n \rightarrow +\infty} u_n = u$ in $L^1(\Omega)$. Condition (73) then yields

$$\lim_{n \rightarrow +\infty} \int_{\Omega} h_{\lambda,\mu}(u_n)(u_n - u) \, dx = 0.$$

We also have

$$\lim_{n \rightarrow +\infty} a(u, u_n - u) = \lim_{n \rightarrow +\infty} \left(\mathcal{K}'_{\lambda,\mu}(u)(u_n - u) + \lambda \int_{\Omega} h_{\lambda,\mu}(u)(u_n - u) \, dx \right) = 0.$$

Indeed, as $\mathcal{K}'_{\lambda,\mu}(u) \in H^{-1}(\Omega)$ and $\lim_{n \rightarrow +\infty} u_n = u$ weakly in $H_0^1(\Omega)$, we see that

$$\lim_{n \rightarrow +\infty} \mathcal{K}'_{\lambda,\mu}(u)(u_n - u) = 0.$$

Finally, as $h_{\lambda,\mu}(u) \in L^\infty(\Omega)$ and $\lim_{n \rightarrow +\infty} u_n = u$ in $L^1(\Omega)$, it follows that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} h_{\lambda,\mu}(u)(u_n - u) dx = 0.$$

This proves that $\mathcal{K}_{\lambda,\mu}$ satisfies the Palais–Smale condition.

The existence of a critical point $u_{\lambda,\mu}^{(3)} \in H_0^1(\Omega)$ of $\mathcal{K}_{\lambda,\mu}$, with $\mathcal{K}_{\lambda,\mu}(u_{\lambda,\mu}^{(3)}) > 0$, for any given $\lambda \in]0, \lambda_0[$ and $\mu \in]\mu_*(\lambda), +\infty[$, then follows from the mountain pass theorem (see e.g. [14, Theorem 5.7]).

Step 5. There exists $\lambda^ > 0$ such that, if $u_{\lambda,\mu}$ is a solution of (75) for some $\lambda \in]0, \lambda^*[$ and $\mu \in]\mu_0, +\infty[$, then $u_{\lambda,\mu} \in C^{1,\tau}(\bar{\Omega})$ for some $\tau \in]0, 1[$, $u_{\lambda,\mu}(x) > 0$ for every $x \in \Omega$, $\frac{\partial u_{\lambda,\mu}}{\partial \nu}(x) < 0$ for every $x \in \partial\Omega$, ν being the unit outer normal to Ω at $x \in \partial\Omega$, and*

$$\|u_{\lambda,\mu}\|_{C^1(\bar{\Omega})} \leq 1. \quad (79)$$

Let $u_{\lambda,\mu}$ be a critical point of $\mathcal{K}_{\lambda,\mu}$ for some $\lambda \in]0, \lambda_0[$ and $\mu \in]\mu_0, +\infty[$. Recall that $u_{\lambda,\mu}$ satisfies (76). Testing (76) against $(u_{\lambda,\mu} - 2)^+$, which belongs to $H_0^1(\Omega)$ by Stampacchia theorem (see [44, Section 1.8]), and using (22) and (72), we get

$$\begin{aligned} \frac{7\sqrt{2}}{16} \int_{\Omega} |\nabla(u_{\lambda,\mu} - 2)^+|^2 dx &\leq \int_{\Omega} a(|\nabla u_{\lambda,\mu}|^2) |\nabla(u_{\lambda,\mu} - 2)^+|^2 dx \\ &= \int_{\Omega} a(|\nabla u_{\lambda,\mu}|^2) \nabla u_{\lambda,\mu} \nabla(u_{\lambda,\mu} - 2)^+ dx \\ &= \lambda \int_{\Omega} \chi(u_{\lambda,\mu}) g_{\lambda,\mu}(u_{\lambda,\mu})(u_{\lambda,\mu} - 2)^+ dx = 0. \end{aligned}$$

Therefore we have $(u_{\lambda,\mu} - 2)^+ = 0$, i.e. $u_{\lambda,\mu}(x) \leq 2$ a.e. in Ω . Testing (76) against $-u_{\lambda,\mu}^-$ and using (22) and (72), we obtain

$$\begin{aligned} \frac{7\sqrt{2}}{16} \int_{\Omega} |\nabla u_{\lambda,\mu}^-|^2 dx &\leq \int_{\Omega} a(|\nabla u_{\lambda,\mu}^-|^2) |\nabla u_{\lambda,\mu}^-|^2 dx \\ &= - \int_{\Omega} a(|\nabla u_{\lambda,\mu}^-|^2) \nabla u_{\lambda,\mu} \nabla u_{\lambda,\mu}^- dx \\ &= -\lambda \int_{\Omega} \chi(u_{\lambda,\mu}) g_{\lambda,\mu}(u_{\lambda,\mu}) u_{\lambda,\mu}^- dx = 0. \end{aligned}$$

Therefore we have $u_{\lambda,\mu}^- = 0$, i.e. $u_{\lambda,\mu}(x) \geq 0$ a.e. in Ω . Thus we conclude that

$$0 \leq u_{\lambda,\mu}(x) \leq 2 \quad (80)$$

for a.e. $x \in \Omega$. Due to (80) and (73), the regularity theory for (75) (see [18]) yields the existence of $\tau \in]0, 1[$ and $K > 0$, independent of $u_{\lambda,\mu}$, such that

$$\|u_{\lambda,\mu}\|_{C^{1,\tau}(\bar{\Omega})} \leq K. \quad (81)$$

Moreover, if $u_{\lambda,\mu} \neq 0$, the strong maximum principle and the boundary point lemma [41, Corollary 8.3, Corollary 8.4] imply that $u_{\lambda,\mu}(x) > 0$ for every $x \in \Omega$ and $\frac{\partial u_{\lambda,\mu}}{\partial \nu}(x) < 0$ for every $x \in \partial\Omega$, where ν is the unit outer normal to Ω at $x \in \partial\Omega$.

Finally, we prove that there exists $\lambda^* \in]0, \lambda_0[$ such that, for every $\lambda \in]0, \lambda^*[$ and $\mu \in]\mu_0, +\infty[$, $u_{\lambda,\mu}$ satisfies (79). Let $(u_{\lambda_n, \mu_n})_n$ be a sequence of solutions of (75) corresponding to some sequences $(\lambda_n)_n \subset]0, \lambda_0[$, and $(\mu_n)_n \subset]\mu_0, +\infty[$. Assume that $\lim_{n \rightarrow +\infty} \lambda_n = 0$. Estimate (81) and the Arzelà–Ascoli theorem yield the existence of a subsequence $(u_k)_k = (u_{\lambda_{n_k}, \mu_{n_k}})_k$ converging in $C^1(\bar{\Omega})$ to some function $u \in C^1(\bar{\Omega})$ with $u(x) = 0$ on $\partial\Omega$. Testing (76) against u_k and using (22), (73) and (80), we get

$$\begin{aligned} \frac{7\sqrt{2}}{16} \int_{\Omega} |\nabla u_k|^2 dx &\leq \int_{\Omega} a(|\nabla u_k|^2) |\nabla u_k|^2 dx \\ &= \lambda_{n_k} \int_{\Omega} h_{\lambda_{n_k}, \mu_{n_k}}(u_k) u_k dx \leq \lambda_{n_k} 2^p \text{meas}(\Omega) \end{aligned}$$

and hence, passing to the limit, $u = 0$. Therefore we deduce that

$$\lim_{\lambda \rightarrow 0^+} \|u_{\lambda,\mu}\|_{C^1(\bar{\Omega})} = 0.$$

Using Step 1, Step 3, Step 4 and Step 5, we conclude that there exists $\lambda^* \in]0, +\infty[$ such that, for any $\lambda \in]0, \lambda^*[$ and $\mu \in]\mu_*(\lambda), +\infty[$, problem (71) has at least three positive solutions $u_{\lambda,\mu}^{(1)}, u_{\lambda,\mu}^{(2)}, u_{\lambda,\mu}^{(3)}$, with, for $i = 2, 3$, $u_{\lambda,\mu}^{(i)} \in C^{1,\tau}(\bar{\Omega})$ for some $\tau \in]0, 1[$, $u_{\lambda,\mu}^{(i)}(x) > 0$ for every $x \in \Omega$ and $\frac{\partial u_{\lambda,\mu}^{(i)}}{\partial \nu}(x) < 0$ for every $x \in \partial\Omega$. \square

3.4. Existence of infinitely many positive solutions

In this section we deal with cases where the potential is neither subquadratic nor superquadratic at zero and it is neither sublinear nor superlinear at infinity, but it oscillates in between. In this frame we can establish the existence of infinitely many positive solutions. The proof combines the lower and upper solutions method, local minimization and critical values estimates, and exploits some ideas from [38,21,36] too.

3.4.1. Potential oscillatory at zero

Theorem 3.15. Assume

- (h₄) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{1,\sigma}$ boundary $\partial\Omega$ for some $\sigma \in]0, 1[$;
- (h₂) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions;
- (h₅) $f(x, 0) \geq 0$ for a.e. $x \in \Omega$;
- (h₆) there exist constants $r > 0$ and $c > 0$ such that $|f(x, s)| \leq c$ for a.e. $x \in \Omega$ and every $s \in [0, r]$;
- (h₂₉) there exist a constant $r > 0$ and a continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x, s) \leq h(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in [0, r]$$

and

$$\liminf_{s \rightarrow 0^+} \frac{H(s)}{s^2} \leq 0,$$

where $H(s) = \int_0^s h(t) dt$;

(h₇) there exist open sets ω and ω_1 , with $\bar{\omega} \subset \omega_1 \subseteq \Omega$, such that

$$\limsup_{s \rightarrow 0^+} \frac{\int_{\omega} F(x, s) dx}{s^2} = +\infty$$

and

$$\liminf_{s \rightarrow 0^+} \frac{\int_{\omega_1 \setminus \omega} F(x, s) dx}{s^2} > -\infty.$$

Then, for every $\lambda > 0$, problem (1) has an infinite sequence $(u_n)_n$ of weak solutions, with $u_n \in C^{1,\tau}(\bar{\Omega})$ for some $\tau \in]0, 1]$, satisfying $u_n(x) > 0$ for every $x \in \Omega$ and $\frac{\partial u_n}{\partial \nu}(x) < 0$ for every $x \in \partial\Omega$, ν being the unit outer normal to Ω at $x \in \partial\Omega$, and

$$\lim_{n \rightarrow +\infty} \|u_n\|_{C^1(\bar{\Omega})} = 0.$$

Proof. Fix $\lambda > 0$. Let us consider, as in the proof of Theorem 3.1, the modified problem (27), where a and g are respectively defined by (20) and (24). By a lower solution of (27) we mean a function $\alpha \in C^1(\bar{\Omega})$ such that $\alpha(x) \leq 0$ on $\partial\Omega$ and

$$\int_{\Omega} a(|\nabla \alpha|^2) \nabla \alpha \nabla v dx \leq \lambda \int_{\Omega} g(x, \alpha) v dx$$

for every $v \in H_0^1(\Omega)$, with $v(x) \geq 0$ a.e. in Ω . An upper solution β of (27) is defined similarly by reversing the first two inequalities written above. It is a known fact (see e.g. [38, Lemma 2.1]) that if there are a lower solution α and an upper solution β of (27), with $\alpha(x) \leq \beta(x)$ in Ω , then there exists at least one solution u of (27) such that $u \in C^{1,\tau}(\bar{\Omega})$ for some $\tau \in]0, 1]$, $\alpha(x) \leq u(x) \leq \beta(x)$ in Ω and

$$\mathcal{K}_{\lambda}(u) = \min \{ \mathcal{K}_{\lambda}(v) \mid v \in H_0^1(\Omega), \alpha(x) \leq v(x) \leq \beta(x) \text{ a.e. in } \Omega \},$$

where \mathcal{K}_{λ} is the functional associated with (27) as defined in (29).

Step 1. There exists a sequence $(\beta_n)_n$ of upper solutions of (27) satisfying $\beta_n \in C^2(\bar{\Omega})$ and $\beta_n(x) > 0$ in Ω , for every n , and $\lim_{n \rightarrow +\infty} \|\beta_n\|_{L^{\infty}(\Omega)} = 0$. Suppose first that $\inf\{s > 0 \mid h(s) \leq 0\} = 0$. Then there exists a sequence of positive constant upper solutions $(\beta_n)_n$ with $\lim_{n \rightarrow +\infty} \beta_n = 0$. Suppose next that there is $r_1 \in]0, r[$ such that $h(s) > 0$ for each $s \in]0, r_1]$. Without loss of generality, we also assume $h(s) \geq 0$ for $s \leq 0$. By (h₂₉) there exists $K \in]0, \lambda_1^*[$, where λ_1^* is defined in (14), such that

$$\liminf_{s \rightarrow 0^+} \frac{2\lambda H(s) - Ks^2}{s^2} < 0.$$

Therefore we can find a decreasing sequence $(R_n)_n$ such that $\lim_{n \rightarrow +\infty} R_n = 0$ and, for each n , $R_n \in]0, r_1[$,

$$\lambda H(R_n) < 1 - \frac{1}{\sqrt{2}}, \quad (82)$$

$$\frac{1}{2} K R_n^2 < 1, \quad (83)$$

and

$$\lambda(H(R_n) - H(s)) \leq \frac{1}{2}K(R_n^2 - s^2) \quad (84)$$

for every $s \in [0, R_n]$. Fix n and consider the initial value problem

$$-(v'/\sqrt{1+|v'|^2})' = \lambda h(v), \quad v(0) = R_n, \quad v'(0) = 0. \quad (85)$$

Let $v \in C^2(]-\omega, \omega[)$ be an even non-extendible solution of (85). Then $v|_{[0, \omega[}$ is decreasing, concave and satisfies the energy relation

$$1 - \frac{1}{\sqrt{1+|v'(t)|^2}} = \lambda(H(R_n) - H(v(t))) \quad (86)$$

in $[0, \omega[$. Define

$$T = \sup\{t \in [0, \omega[\mid v(t) > 0\}.$$

Since $v|_{[0, \omega[}$ is decreasing and concave, we have $T < +\infty$. Then we set

$$v(T) = \lim_{t \rightarrow T} v(t) \geq 0.$$

From (82) and (86) we also get

$$|v'(t)| \leq 1$$

for every $t \in [-T, T]$. Using (86), (84), (83) and the fact that the function $\chi(t) = \frac{1-t}{\sqrt{2-t}}$ is decreasing in $[0, 1]$, we get

$$\begin{aligned} T &= \int_0^T -v'(t) \frac{1 - \lambda(H(R_n) - H(v(t)))}{\sqrt{2\lambda(H(R_n) - H(v(t))) - (\lambda(H(R_n) - H(v(t))))^2}} dt \\ &= \int_{v(T)}^{v(0)} \frac{1}{\sqrt{\lambda(H(R_n) - H(s))}} \chi(\lambda(H(R_n) - H(s))) ds \\ &\geq \int_0^{R_n} \frac{1}{\sqrt{\frac{K}{2}(R_n^2 - s^2)}} \chi\left(\frac{K}{2}(R_n^2 - s^2)\right) ds \\ &\geq \chi\left(\frac{K}{2}R_n^2\right) \sqrt{\frac{2}{K}} \int_0^{R_n} \frac{1}{\sqrt{R_n^2 - s^2}} ds = \chi\left(\frac{K}{2}R_n^2\right) \frac{1}{\sqrt{2}} \frac{\pi}{\sqrt{K}}. \end{aligned} \quad (87)$$

Let $\hat{e} \in S^{N-1}$ be such that $L_{\hat{e}}(\Omega) = \min_{e \in S^{N-1}} L_e(\Omega)$ and set, for every $x \in \bar{\Omega}$,

$$\beta_n(x) = v\left(x\hat{e} - \frac{1}{2}(a_{\hat{e}}(\Omega) + b_{\hat{e}}(\Omega))\right),$$

where $L_{\hat{e}}(\Omega)$, $a_{\hat{e}}(\Omega)$ and $b_{\hat{e}}(\Omega)$ are defined by (13). As $K < \lambda_1^* = (\frac{\pi}{L_{\hat{e}}(\Omega)})^2$ we have $\frac{\pi}{\sqrt{K}} > L_{\hat{e}}(\Omega)$ and, as $\lim_{n \rightarrow +\infty} \chi(\frac{K}{2}R_n^2) = \chi(0) = \frac{1}{\sqrt{2}}$, we conclude from (87) that, for all n large, $T > \frac{1}{2}L_{\hat{e}}(\Omega)$ and hence $\beta_n(x) > 0$ for every $x \in \Omega$. Note that $\beta_n \in C^2(\bar{\Omega})$, $|\nabla \beta_n(x)| = |v'(x\hat{e} - \frac{1}{2}(a_{\hat{e}}(\Omega) + b_{\hat{e}}(\Omega)))| \leq 1$ in Ω and

$$\begin{aligned} -\operatorname{div}(a(|\nabla \beta_n|^2)\nabla \beta_n) &= -\operatorname{div}(\nabla \beta_n / \sqrt{1 + |\nabla \beta_n|^2}) \\ &= -v'' / (1 + |v'|^2)^{\frac{3}{2}} = \lambda h(v) \geq \lambda g(x, \beta_n) \end{aligned}$$

a.e. in Ω . Therefore β_n is an upper solution of (27). Further we have $\lim_{n \rightarrow +\infty} \|\beta_n\|_{L^\infty(\Omega)} = 0$.

Step 2. Existence of solutions of (1). Condition (h_5) implies that $\alpha = 0$ is a lower solution of (27). Hence there exists a solution $u_1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (27) such that $0 \leq u_1(x) \leq \beta_1(x)$ in Ω and

$$\mathcal{K}_\lambda(u_1) = \min\{\mathcal{K}_\lambda(v) \mid v \in H_0^1(\Omega), 0 \leq v(x) \leq \beta_1(x) \text{ a.e. in } \Omega\}.$$

Arguing as in Step 2 of the proof of Theorem 3.1, we exploit assumption (h_7) to show that $\mathcal{K}_\lambda(u_1) < 0$ and, hence, $u_1 \neq 0$. Next we pick an upper solution β_2 of (27) such that $\|\beta_2\|_{L^\infty(\Omega)} < \|u_1\|_{L^\infty(\Omega)}$. Proceeding as above we find a solution $u_2 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (27) such that $0 \leq u_2(x) \leq \beta_2(x)$ in Ω and $u_2 \neq 0$. Iterating this argument we obtain a sequence of non-trivial non-negative solutions of (27) such that

$$\lim_{n \rightarrow +\infty} \|u_n\|_{L^\infty(\Omega)} = 0.$$

Using (25), which follows from (h_6) , and the regularity theory for (27) (see [18]), we infer that there are constants $\tau \in]0, 1[$ and $\kappa > 0$ such that, for every n , $u_n \in H_0^1(\Omega) \cap C^{1,\tau}(\bar{\Omega})$ and

$$\|u_n\|_{C^{1,\tau}(\bar{\Omega})} \leq \kappa.$$

Hence, by the Arzelà–Ascoli theorem we deduce that

$$\lim_{n \rightarrow +\infty} \|u_n\|_{C^1(\bar{\Omega})} = 0.$$

Accordingly we conclude that, for all n sufficiently large, u_n is a positive weak solution of (1). The strong maximum principle and the boundary point lemma [41, Corollary 8.3, Corollary 8.4] finally yield $u_n(x) > 0$ for every $x \in \Omega$ and $\frac{\partial u_n}{\partial \nu}(x) < 0$ for every $x \in \partial\Omega$. \square

The following result shows that for the parameter independent problem (48) the assumptions (h_7) and (h_{29}) on the oscillatory behavior of F at 0 can be replaced by some conditions involving the spectral constants λ_1^\sharp and λ_1^* .

Theorem 3.16. *Assume*

- (h_4) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{1,\sigma}$ boundary $\partial\Omega$ for some $\sigma \in]0, 1[$;
- (h_2) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions;
- (h_{30}) there exist a constant $r > 0$ and a continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x, s) \leq h(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in [0, r]$$

and

$$\liminf_{s \rightarrow 0^+} \frac{2H(s)}{s^2} < \lambda_1^*,$$

where $H(s) = \int_0^s h(t) dt$ and λ_1^* is defined by (14);

(h₁₈) there exist a constant $r > 0$ and a continuous function $k : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x, s) \geq k(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in [0, r],$$

$$\liminf_{s \rightarrow 0^+} \frac{K(s)}{s^2} \geq 0$$

and

$$\limsup_{s \rightarrow 0^+} \frac{2K(s)}{s^2} > \lambda_1^\sharp,$$

where $K(s) = \int_0^s k(t) dt$ and λ_1^\sharp is defined by (16).

Then problem (48) has an infinite sequence $(u_n)_n$ of weak solutions, with $u_n \in C^{1,\tau}(\bar{\Omega})$ for some $\tau \in]0, 1[$, satisfying $u_n(x) > 0$ for every $x \in \Omega$ and $\frac{\partial u_n}{\partial \nu}(x) < 0$ for every $x \in \partial\Omega$, ν being the unit outer normal to Ω at $x \in \partial\Omega$, and

$$\lim_{n \rightarrow +\infty} \|u_n\|_{C^1(\bar{\Omega})} = 0.$$

Proof. The conclusion follows arguing like in the proofs of Theorem 3.15 and Theorem 3.7. \square

3.4.2. Potential oscillatory at infinity

Theorem 3.17. Assume

(h₁) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{0,1}$ boundary $\partial\Omega$;

(h₅) $f(x, 0) \geq 0$ for a.e. $x \in \Omega$;

(h₁₃) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the L^p -Carathéodory conditions for some $p > N$;

(h₁₄) there exist a constant r and a continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x, s) \leq h(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \geq r$$

and

$$\liminf_{s \rightarrow +\infty} \frac{H(s)}{s} \leq 0,$$

where $H(s) = \int_0^s h(t) dt$;

(h₃₁) there exists a Caccioppoli set $B \subseteq \Omega$ such that

$$\limsup_{s \rightarrow +\infty} \int_B \frac{F(x, s)}{s} dx = +\infty.$$

Then, for every $\lambda > 0$, problem (1) has an infinite sequence $(u_n)_n$ of positive solutions, with $u_n \in L^\infty(\Omega)$, satisfying

$$\lim_{n \rightarrow +\infty} \|u_n\|_{L^\infty(\Omega)} = +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} \mathcal{I}_\lambda(u_n) = -\infty.$$

Proof. Fix $\lambda > 0$. Conditions (h_5) and (h_{13}) imply that $\alpha = 0$ is a lower solution of (1). By (h_{14}) we can apply the claim in the proof of Theorem 3.3 to get a sequence $(\beta_n)_n$ of upper solutions of (1) such that, for each n , $\beta_n \in C^2(\bar{\Omega})$ and $\lim_{n \rightarrow +\infty} (\min_{\bar{\Omega}} \beta_n) = +\infty$. By (h_{31}) there is a sequence $(c_n)_n$ such that $\lim_{n \rightarrow +\infty} c_n = +\infty$ and

$$\lim_{n \rightarrow +\infty} \int_B \frac{\lambda F(x, c_n)}{c_n} dx > \text{Per}(B),$$

where $\text{Per}(B) = \int_{\Omega} |D\chi_B| + \int_{\partial\Omega} \chi_B |_{\partial\Omega} d\mathcal{H}_{N-1}$ is the perimeter of B in \mathbb{R}^N (χ_B denoting the characteristic function of the set B). We have

$$\begin{aligned} \mathcal{I}_\lambda(c_n \chi_B) &= \mathcal{J}(c_n \chi_B) - \int_B \lambda F(x, c_n) dx \\ &\leq \text{meas}(\Omega) + c_n \left(\int_B |D\chi_B| + \int_{\partial\Omega} \chi_B |_{\partial\Omega} d\mathcal{H}_{N-1} - \int_B \frac{\lambda F(x, c_n)}{c_n} dx \right) \\ &= \text{meas}(\Omega) + c_n \left(\text{Per}(B) - \int_B \frac{\lambda F(x, c_n)}{c_n} dx \right) \end{aligned}$$

and hence

$$\lim_{n \rightarrow +\infty} \mathcal{I}_\lambda(c_n \chi_B) = -\infty.$$

Therefore we can find a constant, say c_1 , such that $\mathcal{I}_\lambda(c_1 \chi_B) < \mathcal{I}_\lambda(0)$. Pick an upper solution, say β_1 , such that $\beta_1(x) \geq c_1 \chi_B(x)$ in Ω . By Proposition 2.1 there exists a solution u_1 of (1) such that $0 \leq u_1(x) \leq \beta_1(x)$ a.e. in Ω . Moreover, we have $u_1 \neq 0$, as $\mathcal{I}_\lambda(u_1) = \min\{\mathcal{I}_\lambda(v) \mid v \in BV(\Omega), 0 \leq v(x) \leq \beta_1(x) \text{ for a.e. } x \in \Omega\} < \mathcal{I}_\lambda(0)$. Pick now a constant, say c_2 , such that $\mathcal{I}_\lambda(c_2 \chi_B) < \mathcal{I}_\lambda(u_1)$ and an upper solution, say β_2 , such that $\beta_2(x) \geq c_2 \chi_B(x)$ in Ω . By Proposition 2.1 there exists a solution u_2 of (1) such that $0 \leq u_2(x) \leq \beta_2(x)$ a.e. in Ω and $\mathcal{I}_\lambda(u_2) \leq \mathcal{I}_\lambda(c_2 \chi_B) < \mathcal{I}_\lambda(u_1)$. Hence, we have in particular $u_1 \neq u_2$ and $\text{ess sup}_\Omega u_2 > \min_{\bar{\Omega}} \beta_1$. Iterating this procedure we can construct a sequence $(u_n)_n$ of solutions of (1) such that, for each n , $u_n \in BV(\Omega) \cap L^\infty(\Omega)$ and $\mathcal{I}_\lambda(u_{n+1}) \leq \mathcal{I}_\lambda(c_{n+1} \chi_B) < \mathcal{I}_\lambda(u_n)$, $\lim_{n \rightarrow +\infty} \text{ess sup}_\Omega u_n = +\infty$ and $\lim_{n \rightarrow +\infty} \mathcal{I}_\lambda(u_n) = -\infty$. \square

The following result is the counterpart of Theorem 3.16 when an oscillatory behavior of F at $+\infty$ is considered.

Theorem 3.18. Assume

- (h_1) Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a $C^{0,1}$ boundary $\partial\Omega$;
- (h_5) $f(x, 0) \geq 0$ for a.e. $x \in \Omega$;
- (h_{13}) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the L^p -Carathéodory conditions for some $p > N$;
- (h_{19}) there exist a constant $r > 0$ and a continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x, s) \leq h(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \geq r$$

and

$$\liminf_{s \rightarrow +\infty} \frac{H(s)}{s} < \mu_1^*,$$

where $H(s) = \int_0^s h(t) dt$ and μ_1^* is defined by (15);

(h₃₂) there exist a constant $r > 0$ and a continuous function $k : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x, s) \geq k(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \geq r$$

and

$$\limsup_{s \rightarrow +\infty} \frac{K(s)}{s} > \mu_1^\sharp,$$

where $K(s) = \int_0^s k(t) dt$ and μ_1^\sharp is defined by (18).

Then problem (48) has an infinite sequence $(u_n)_n$ of positive solutions, with $u_n \in L^\infty(\Omega)$, satisfying

$$\lim_{n \rightarrow +\infty} \|u_n\|_{L^\infty(\Omega)} = +\infty.$$

Proof. The conclusion follows as in the proof of Theorem 3.17, once we observe that (h₁₉) is sufficient to apply the claim in the proof of Theorem 3.3 and (h₃₂) implies the existence of a sequence $(c_n)_n$ such that $\lim_{n \rightarrow +\infty} c_n = +\infty$ and

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \frac{F(x, c_n)}{c_n} dx > \text{Per}(\Omega). \quad \square$$

Remark 3.7. Assumption (h₃₂) can be replaced by

(h₃₃) there exists a Caccioppoli set $B \subseteq \Omega$ such that

$$\limsup_{s \rightarrow +\infty} \int_B \frac{F(x, s)}{s} dx > \text{Per}(B).$$

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